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BY

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Professor of Astronomy, University of California;
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CELESTIAL MECHANICS

REPORT OF THE COMMITTEE ON CELESTIAL MECHANICS OF THE NATIONAL RESEARCH COUNCIL*

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E. W. BROWN, Professor of Mathematics, Yale University, Chairman; G. D. BIRKHOFF, Professor of Mathematics, Harvard University; A. O. LEUSCHNER, Professor of Astronomy, University of California; H. N. RUSSELL, Professor of Astronomy, Princeton University

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REPORT OF THE COMMITTEE ON CELESTIAL MECHANICS

Celestial mechanics, broadly interpreted, is involved in practically all the astronomy of the present time. The limited meaning of the term now usually adopted refers only to those problems in which the law of gravitation plays the chief or only part, and more particularly to those which deal with the motions of bodies about one another and with their rotations. This limitation will, in any case, be adopted in this report since surveys dealing with other aspects of astronomy have been written or are contemplated. It is, however, necessary not to be too rigid about the border lines, especially in considering questions where the gravitational action does not fully account for the observed phenomena.

The report has three general divisions: I, the solar system; II, the stellar system; III, the theoretical aspects of the general problem of three or more bodies. It is not intended to contain a com-

* The membership of the committee is: E. W. Brown, chairman, G. D. Birkhoff, A. O. Leuschner, H. N. Russell.

plete account of the present status of the subject. More emphasis has been laid on those portions which are under discussion at the present time and on problems, at present unsolved, which need discussion and solution.

PART I. THE SOLAR SYSTEM

The order of treatment is as follows: The moon, the eight major planets, their satellites other than the moon, the asteroids or minor planets, comets. The general view of the subject now as in the past, has been to consider the consequences of the law of gravitation, the extent to which it accounts for the observed motions—leading to the discovery of other possible influences—and prediction for future observation and comparison with theory.

The Moon. The gravitational motion has been worked out sufficiently to satisfy all observational needs of the past and probably of some centuries in the future, and the results are fully embodied in tables constructed to furnish the moon's position without excessive labor. The observational data are the daily Greenwich observations (weather permitting) since 1750, isolated series of observations, eclipses, occultations since the beginning of the sixteenth century, and occasional ancient records of eclipses and occultations during the past forty centuries. These have led to the establishment of the following differences from a purely gravitational theory:

(a) An apparent secular acceleration of the moon's mean motion of about 4″5* per century, per century, combined with an acceleration of the earth's mean motion about the sun ("acceleration of the sun") of a little over 1″ with probable errors, according to Fotheringham,[1] to whom the latest figures are due, of about ±0″.5. The former has frequently been attributed to a slowing down of the earth's rate of rotation due to tidal friction: the new work of G. I. Taylor[2] and H. Jeffreys[3] has rendered this explanation very probable both qualitatively and quantitatively, especially as it also accounts for most of the sun's acceleration.

(b) A long-period term of some 275 years period and 13″ amplitude in the mean longitude, obtained from observations extending

* This is the coefficient of t^2 in the expression for the longitude generally misnamed the acceleration. The true acceleration is therefore twice this amount.

[1] *M. N. R. A. S.*, **80**, p. 581.

[2] *Phil. Trans. R. S.*, **220**, p. 1.

[3] *Ibid.*, **221**, p. 239; *M. N. R. A. S.*, **80**, 309.

over about the same time. Numerous hypotheses have been advanced to account for this deviation, but none of them rest on any secure physical basis nor have they received independent testimony.[1]

(c) Fluctuations which are evident in the observations of the past 170 years and well defined during the last 70 years. In the former time their principal period seems to have decreased from some 70 years to about 40 years, with an amplitude of some 3″ or 4″. In 1914 E. W. Brown[2] pointed out that similar fluctuations of much smaller amplitude could be traced in the motions of the earth and of Mercury: these fluctuations were confirmed by Glauert[3] who found them also in the longitude of Venus. The latter also showed that they could all be moderately well accounted for by changes in the rate of rotation of the earth. No cause is assigned for these changes and their magnitude, amounting sometimes to as much as a loss or gain of $0^s.07$ in one year in the rotation of the earth considered as a clock, makes the acceptance of the hypothesis difficult. It hass been uggested that this hypothesis might be tested by observations of the eclipses of Jupiter's satellites, which at present seem to furnish the only possible means for the purpose.

The Major Planets. The theories and tables by Newcomb and Hill seem to satisfy present needs, except perhaps those of Jupiter and Saturn, into which some small errors have crept but are now in process of examination by Innes. The sufficiency of the adopted theories is well shown by the theoretical and observed secular changes of the perihelia, nodes, eccentricities and inclination. The only large outstanding difference, that of the perihelion of Mercury, is fully accounted for by Einstein's addition to the Newtonian law, although one or two others need to be kept in mind as being perhaps in excess of their actual errors. It may be mentioned that the Einstein addition causes an increase of about 2″ in the centennial motions of the lunar perigee and node[4] but this is just at the limit of accuracy of Brown's theory and probably much beyond detection by observation for many decades to come.

Attempts made to discover a supposed trans-Neptunian planet by its perturbations on Neptune or Uranus[5] have been unsuccessful. The errors of the latter though considerable when the tables of Lever-

[1] See E. W. Brown, *Amer. Jour. Sc.*, Ser. 4, **29**, p. 529.

[2] *Brit. Assoc. Report*, 1914, p. 320.

[3] *M. N. R. A. S.*, **75**, p. 489.

[4] de Sitter, *M. N. R. A. S.*, **77**, p. 172.

[5] P. Lowell, *Lowell Obs. Trans.*, **1**.

rier or Gaillot are used, have become very small with Newcomb's tables, and the observations of Neptune are not sufficient for the purpose.

From time to time, numerical relations of the masses, distances, etc., like those contained in Bode's law, appear but have not so far given theoretical results. A curious fact concerning the distribution of the poles of the planetary orbits, noted by H. C. Plummer,[1] deserves mention.

Satellites. Neither the Uranian nor Neptunian systems present many points of interest to the theoretical astronomer, on account of their distances from the sun and Earth. The four inner satellites of Jupiter, partly on account of the librational relation between three of them and partly because of the possibility of testing the constancy of the rate of rotation of the Earth by observations of their eclipses, have been again considered by R. A. Sampson, who has worked out their theory and has published tables. The outer satellites present several features of mathematical and physical interest. In Hyperion and Titan, satellites of Saturn, there is another case of libration worked out to a limited extent by Newcomb. Since the issue of Vol. IV of Tisserand's Mécanique Celeste in 1896, which contains a full account of the work to that date, the orbit of Phoebe (Saturnian system), and of the sixth satellite of Jupiter have been worked out by Ross.[2]

Asteroids. Nearly one thousand of these bodies are now known. From the gravitational point of view, they possess the greatest mathematical interest, on account of the large perturbations produced in their orbits by Jupiter.

Long period inequalities constitute the chief difficulty in all the gravitational problems of the solar system, on account of the further approximations needed to obtain the required degree of accuracy of the numerical values of the comparatively large coefficients. The older methods went ahead without reference to them and carried out the approximations as they were needed, as for instance in Hansen's method used by G. W. Hill for Jupiter and Saturn in which a very long period term with a large coefficient causes most of the trouble. In the newer methods initiated by Gyldén and his followers, among them Backlund and Brendel, an attempt is made to introduce such terms as early as possible so as to diminish the ultimate work of computation.

[1] *M. N. R A. S.*, 76, p. 378.

[2] *Harvard Annals*, 53, VI.

There are two types of oscillation which differ in their mathematical treatment and in their physical results. The ordinary long-period type is that in which a forced oscillation has a period near that of a free oscillation. But when the two periods become almost exactly the same, the free oscillation is compelled to take the period of the forced oscillation, and there is then a new oscillation of finite period about this position: the latter is called a libration. It is well known that though the periods of the asteroids have a considerable range, there are none certainly known whose periods are exactly $^1/_2$, $^1/_3$, $^2/_5$ that of Jupiter, while there are considerable numbers with periods a little different from these fractions. It is obvious that the resonance has some relation to the distribution, but so far all mathematical investigation has failed to show any reason for the gaps: there is no evidence of instability in the deductions from the equations of motion. Attempts to search for the cause in cosmogonic speculations or in a resisting medium have been made, but a more complete investigation of the gravitational effects is needed. The problem is similar to that of the divisions in Saturn's ring, connected with perturbations produced by the satellites. The question is complicated by the fact that several cases of libration exist without apparent instability, e. g., amongst the satellites of Jupiter, of Saturn, in the Trojan group of asteroids whose mean period is the same as that of Jupiter and best known of all, in the rotation of the moon which has the same period as that of its revolution around the earth.

Numerous statistical investigations have been carried out, but little has been deduced from them, except in the way of confirming known perturbative effects.

Closely related in importance to the foregoing purely theoretical considerations is the practical problem of suitable numerical methods for the representation of the motion of the asteroids. Hansen's and the Gyldén-Brendel methods have been referred to. Of fundamental importance for practical purposes are also the methods inaugurated by Bohlin for the group determination of the perturbations of planets which have a mean motion nearly commensurable with that of Jupiter. Bohlin's method rests on Hansen and has been followed by Von Zeipel, Leuschner, and D. F. Wilson, by the former in application to the group $^1/_2$, by the latter to the group $^2/_5$. Bohlin's developments are general for all groups, with special application to the group $^1/_3$. A feature of these methods is the use of elements which are similar to the elements ordinarily known as mean elements. In all of these methods the mean motion, eccentricity, and inclination may lead to complications. The success of any method depends upon the possibility of meeting the complications arising from critical values of these three elements separately and jointly. This has not been attempted in practice to a degree of precision which would reveal any departures from the motion under the Newtonian law of the type of the motion of the perihelion of Mercury, although for planets with moderate inclination and eccentricity and a mean motion not nearly commensurable with that of Jupiter, Hansen's method appears entirely suitable.

The principal aim of astronomers at the present time is to represent the motions with sufficient approximation to serve purposes of identification and observation. Even with this limitation of accuracy the difficulties are considerable. Leuschner published preliminary results of his experience with the Watson asteroids.[1] Unpublished later results of the planets 10 Hygiea and 175 Andromache verify his conclusions that the revised tables of von Zeipel for the group ½ will give the most satisfactory results for all known planets of this group. It would seem therefore extremely advisable to have tables computed on Bohlin's or similar plans for other than the three groups for which they are available. The Trojan group, however, does not appear to lend itself to treatment by Bohlin's method without certain modifications. The perturbations of this group are being successfully dealt with by E. W. Brown, and Wilkins[2] has represented the observations of 884 Priamus by his own treatment to within 10″ of arc from one opposition to the next by including second order perturbations of the first degree in the eccentricity, inclination, and deviation from the center of libration. This method will more than answer practical requirements from one opposition to the next, while Brown's developments are intended to represent positions at any time.

Brendel classifies the planets after Gyldén as ordinary, characteristic and critical planets according to the ratio of their mean motions to that of Jupiter, the critical planets being those of close commensurability. Application of his method has been made by Brendel to one hundred planets with mean motions from 800 to 852. The elements are instantaneous but not osculating and perturbations greater than 3′.4 within fifty years are included so as to reproduce the geocentric places within 20′ for one hundred years.

[1] Proceedings of the National Academy of Sciences, 5, pp. 67–76, March, 1919.

[2] *A. N.* 208, 233.

The secular perturbations in the longitude of the perihelion and of the node are the same for all these planets. The periodic perturbations of the various elements adopted are computed with the aid of five to nine constants for each planet and four arguments, linear with the time. Later Brendel[1] has published improved elements for sixty of these planets and added three to the list.

Among other applications of Brendel's method are those by Labitzke[1] who has published mean elements and approximate perturbations by Jupiter for nineteen of the planets with mean motions between 780″ and 857″ with a somewhat lower degree of approximation than that aimed at by Brendel for the previous list. Boda[1] has published approximate perturbations for one hundred and eight planets of the group $^1/_3$, Hestia group, with mean motions ranging from 845″ to 958″, in two groups according to whether the mean motion is greater or less than 897″. The perturbations are not very large in these cases because the longitudes of perihelia are near those of Jupiter. These approximations by Brendel's method serve a very useful purpose.

Among other investigations which may serve for comparison of the suitability of various methods are those by Notebaum[2] and Osten.[3] The former has developed the perturbations of 433 Eros after Leverrier. Commensurability with the mean motions of the three perturbing planets is not involved. The latter has developed the perturbations of the third order due to two major planets for 447 Valentine by Hansen's method, reproducing the normal places within ± 1″ to 2″ over 19 years. A preliminary review of the present status of the determination of the perturbations of minor planets is in progress under the direction of Leuschner.

Of interest in connection with the question of stability for commensurable planets are certain numerical results. Brown's studies and von Zeipel's developments do not indicate instability. Miss Levy's unpublished developments by the von Zeipel method place the mean mean motion of Andromache at 619.5″ and according to Berberich's computations by the method of special perturbations the osculating value of the mean motion has diminished with slight periodic variations from 617.7″ in 1877 to 607.8″ in 1921, so that the

[1] Brendel, *A. N.* 195, 417; *A. N.*, 200, 1; Labitzke, *A. N.*, 212, 217; Boda, *A. N.* 212, 219.

[2] *A. N.*, 214, 153.

[3] *Astr. Abh.*, 15; *A. N.*, 210, 130.

hitherto accepted gap has been well invaded in this case. On the other hand Krassowski comes to the conclusion on the basis of Brendel's theory that for planets of mean motion near 400″ or $^4/_3$ times Jupiter's mean motion, the motion of an asteroid would become unstable between the limits 391.5″ and 401.7″. (Travaux de la Société des Sciences de Varsovie III. Classe des sciences math. et. nat. Nr. 12.)

Comets. The principal question of interest regarding comets is that of their origin, whether they are members of the solar system with elliptic orbits or enter it from without in parabolic or hyperbolic orbits. The possibility of the "capture" of a comet by diversion from a parabolic into an elliptic orbit has been recognized since the days of Laplace and discussed in much detail by H. A. Newton.[1] Leuschner[2] has shown that the numerous "parabolic" orbits which appear in catalogues of comets, represent cases in which the observations are insufficient to detect the deviations from a parabola which almost always appear when the observations are sufficiently numerous and cover a long enough interval. In the latter cases the orbit is usually definitely elliptic. The osculating orbit near perihelion is occasionally hyperbolic, but Fayet[3] and Strömgren[4] have shown that in every case of this sort the approach has been in an elliptic orbit which was later converted to a hyperbolic by planetary perturbations.

All comets so far recorded appear, therefore, to be members of the solar system. The origin of the cometary orbits of shorter period, especially of the numerous group having a period between four and eight years, has been attributed to capture of comets of longer period by encounters with the major planets. All investigators agree in confirming this explanation in the case of the "Jupiter family" described above. Russell[5] has shown that there is very little evidence of such capture among the comets of periods between 10 and 2,000 years, and that the supposed "families" of Saturn, Uranus, and Neptune have little or no foundation. Their large number is probably due to the disruption of an originally smaller number of comets by close approach to Jupiter as suggested

[1] *Memoirs Nat. Acad. Sciences*, **6**, 7–23, 1893.

[2] *Pub. A. S. P.*, **19**, 67–71, 1907.

[3] Annales de l'observatoire de Paris, Mémories, **26**, 1910.

[4] *Pub. fra Kobenhavn's Observatorium*, **19**, 61, 1914.

[5] *A. J.*, **33**, 49–61, 1920. (This paper contains a number of references to earlier works.)

by Callandreau[1] and several groups of comets of probably common origin have been indicated by Fayet.[2]

These comets may have had substantially their present periods for an indefinite period or have attained them by the slow summation of small perturbations of the ordinary sort.

The motion of the matter in the tails of comets, which observations show to be repelled by the Sun, and probably also by the comet's head,[3] also presents problems of considerable interest, though the unavoidable lack of precision in the observations makes exact computations difficult.

PART II. CELESTIAL MECHANICS AS APPLIED TO THE STARS

The incidence of the application of mathematical analysis to sidereal problems is quite different from that which is found within the solar system. The determination of orbits and disturbed motion here take a subordinate place, and are largely supplanted by questions relating to the statistical equilibrium of systems of great numbers of stars, to the internal constitution, rotation, and vibrations of gaseous masses, and to theories of the evolution of the Universe and its separate portions. Throughout this field the dynamical discussion must keep in close touch with statistical methods and, above all, with physics, especially atomic physics.

1. The problem of the determination of orbits, whether of visual double stars or of spectroscopic or eclipsing binaries, differs radically from that presented by the orbit of a planet, comet or satellite, because of the great difference in the percentage accuracy of the observations. Errors amounting to ten per cent of the observed quantity are habitually met with, and it is, therefore, hopeless to attempt to derive reliable elements from the theoretical minimum number of data.

Only when numerous observations, covering at least a considerable fraction of the period, are available, is it worth while to compute; and graphical methods, based upon curves drawn to represent the whole course of the observations, are universally employed —though the later treatment is often in part analytical. The number of existing methods of solution is considerable, and their practical application is an art, fully as much as a science. Preference should be given to those procedures which enable the computer to

[1] *Annales de l'Obs. de Paris*, **22**, p. 1–47, 1902.

[2] *Bull. Ast.*, **28**, 170, 1911.

[3] *Eddington M. M.*, **70**, 442–458, 1910.

keep closest to the original observations, rather than to the empirical curves which have been drawn to represent them.

An excellent detailed discussion of all three sorts of binary stars will be found in Aitken's recent volume.[1]

2. In view of these facts, it is not surprising that relatively little work has been done upon perturbations in multiple stellar systems, where orbital motion is shown. Such systems, so far as is known, always exhibit a close pair, attended at a relatively considerable distance by a companion (sometimes itself double) revolving in an orbit of much longer period. In a few cases[2] the system thus formed has itself a remote attendant, presumably in very slow orbital motion.

The masses of all the components, so far as they are ascertainable, are of the same order of magnitude. The problems thus presented are analogous to those of the Lunar, rather than the Planetary Theory, but are complicated by the great eccentricities of the orbits, which sometimes exceed 0.7.

There is, however, not as yet a single system in which the orbital elements of both the close and the wide pair are fully known. When the close pair is telescopically separable, the period of the distant companion is so long that it will be several centuries before a reliable orbit can be computed. Seeliger[3] has discussed the perturbations in such systems. The cases in which one or both components of a visual binary are themselves spectroscopic binaries of short period are more promising, but certain elements (notably the inclination of the orbit plane) cannot be found from the spectroscopic observations. Two such systems[4,5] have been observed long enough to show definite evidences of perturbations of the close system (advance of the line of apsides and changes in period) and it is very desirable that they should be studied analytically.

3. The distribution and motions of the stars in space, and in particular the distribution of velocities which is known as "Star Streaming" afford an attractive field for dynamical study. Here we are concerned with the statistical distribution of the coordinates

[1] R. G. Aitken, *The Binary Stars* (New York, 1918) Chapters 4, 6 and 7.

[2] *α Geminorum and ε Hydrae.*

[3] H. Seeliger, *Abhandlungen de Münchener Akad.*, II Kl, 17, 1011 (1888) (ζ Cancri) *Astronomische Nachrichten*, 173, 327 (1907) (ε Hydrae).

[4] κ. Pegasi; F. Henroteau, *Lick Observatory Bulletin*, 9, 120 (1918). Period of close pair 5.97 days; of wide pair 11.35 years.

[5] 13 Ceti; J. S. Paraskevopoulos, *Astrophysical Journal*, 52, 110 (1920). Short period 2.08 days; long period 6.88 years.

and velocities of an enormous number of bodies under their mutual gravitation, and the analysis resembles in many ways that employed in the kinetic theory of gases.[1]

The stars are, however, so small in comparison with the distances between them that the influence of collisions, or even of encounters such that their mutual attraction changes their directions of motion by a degree, may be neglected, unless enormously long intervals of time are involved. The "time of relaxation" which would be required to produce extensive alterations in the velocities of the stars by such encounters is estimated by Jeans[2] as 10^{14} years and by Charlier[3] as 10^{16} years.

The motions of the stars under the general attraction of the whole mass of stars have been discussed by Eddington[4] and Jeans.[5] They find that a steady state in which star-streaming occurs is possible with a spherical "universe" and the direction of streaming radial, or with a "universe" shaped like a figure of revolution, and with streaming taking place along circles coaxial with the axis of symmetry. The first of these models is unlike the actual universe of stars, while the applicability of the second is uncertain. It is, however, very doubtful whether our universe is in a steady state—especially in view of the enormous time which would be required to reach one. Jeans[6] has shown that a similar type of streaming might be produced as the result of successive encounters of our "universe" with other star clusters, but in a later discussion[7] he reverts tentatively to the previous hypothesis.

In the globular clusters, the motions are unknown, but the distribution of the stars in space is remarkably similar from cluster to cluster, and follows closely the law $\rho = \rho_0(1 + r^2/a^2)^{-5/2}$ (where ρ is the density of distribution, r the distance from the centre, and ρ_0 and a are constants). Jeans[8] and Eddington[9] have discussed this question. They find suggestions that this distribution may represent the nearest approach to a state of equipartition of energy

[1] Summarized by Jeans, "Problems of Cosmogony" (Cambridge University Press, 1919) Chapter X.

[2] J. H. Jeans, *M. N. R. A. S.*, **74**, 112 (1913).

[3] C. V. L. Charlier, *Meddelanden från Lunds Astron. Observatorium*, Series II, No. 16, 84 (1917).

[4] A. S. Eddington, *M. N. R. A. S.*, **74**, 5 (1913); **75**, 366 (1915), and **76**, 37 (1915).

[5] J. H. Jeans, *M. N. R. A. S.*, **76**, 70 (1915).

[6] J. H. Jeans, *M. N. R. A. S.*, **76**, 552 (1916).

[7] "Problems of Cosmogony," pp. 236–242.

[8] *M. N. R. A. S.*, **76**, 567 (1916); "Problems of Cosmogony," p. 245.

[9] *M. N. R. A. S.*, **76**, 572 (1916).

which is possible without a scattering of the outer stars to infinity; but the problem is by no means yet solved.

4. The modern theory of the internal constitution of the stars begins with the work of Schwarzschild[1] who called attention to the importance of "radiative equilibrium." The transfer of heat outward from the interior takes place almost entirely by the emission of radiation and its absorption in overlying layers, and the temperature gradient is determined by the outgoing flux of heat and the opacity of the material to the radiation. Schwarzschild dealt only with the atmosphere of the Sun, but Eddington[2] extended the analysis to the interior of the stars, both those of low density, within which the simple gas laws are obeyed at all points, and later[3] to those of higher density. He was the first to point out the importance of radiation pressure, which at the very high temperatures that prevail inside the stars becomes great enough to counteract a large part of the gravitational force, and of ionization, which breaks up most of the atoms into nuclei (or small nuclear groups) and free electrons, and greatly reduces the molecular weight. Upon certain plausible assumptions, he concluded that the ratio of the radiation pressure to the total pressure is constant throughout the star. Hence, the gas pressure is proportional to the fourth power of the temperature—a relation which suffices to define the whole internal constitution of the star, when combined with the law connecting temperature, pressure, and density, for which in general, Eddington adopts a simplified form of Van der Waals' Law.

If β is the ratio of the gas pressure to the total pressure it follows that $(1 - \beta) / \beta^4 = CM^2m^4$ where M is the mass of the star, *m* the mean molecular weight of the material composing it, and C is a constant, which is the same for all stars of low density, and depends only on fundamental physical constants of gravitation, radiation and gas-theory.

For bodies of mass less than about 3×10^{32} grams, $(1 - \beta)/\beta^4$ is small, and the radiation pressure is almost negligible. For masses greater than 3×10^{34} grams $(1 - \beta)/\beta^4$ is large, and the radiation pressure almost neutralizes gravitation, and is the dominant influence in the internal equilibrium. The interval within which the change takes place is exactly that in which all known stellar masses lie. The masses of the stars appear, therefore, to be deter-

[1] *Göttingen Nachrichten*, 1906, 141.

[2] *M. N. R. A. S.*, 77, 16 (1916).

[3] *M. N. R. A. S.*, 77, 596 (1917).

mined by the fundamental properties of atoms. Bodies of smaller mass do not radiate enough to be visible as stars, while those of greater mass are in such a delicate state of equilibrium that they tend to break up into smaller masses.

When the mean density increases, and departures from the simple laws must be considered, the constant C decreases rapidly. Eddington has determined it by quadratures.

The total radiation from a star's surface is proportional to $M(1 - \beta)/K$ where K is the coefficient of opacity of the material. As the star contracts, the radiation will be relatively great, and nearly constant, till the density becomes considerable, and will then fall steadily.

Certain of Eddington's assumptions have been severely criticized by Jeans,[1] who proposes a modified theory.[2] The principal point at issue is the constancy of the ratio of radiation pressure to gas pressure throughout the star. This will be constant, if, and only if, the product $K\eta$ is constant, where K is the opacity of the material, and η the mean rate of generation of heat, per unit time, per unit mass, in the portion of the star nearer the center than the region considered. It is very probable that η increases toward the center and that K decreases. Eddington, in his original discussion, took them as separately constant; but this is unnecessary. He does not present his results as rigorous solutions of the physical problem, but as solutions of a simplified problem analogous to the more complicated reality. The manner in which his "model" agrees with the known properties of the stars is so striking as fully to justify his contention that the assumptions on which it is based are probably not far from the truth.

Jeans has further shown that, if the energy of the stars is derived from gravitational contraction, those of moderate mass would develop more rapidly than those of either small or great mass.

5. The oscillations of a gaseous star about its normal equilibrium have been discussed by Moulton[3] and Eddington[4] with a view to the explanation of the variation of stars of the short period or Cepheid type. Moulton's conclusion that a very small oscillation from a prolate to an oblate spheroidal form would account for great

[1] *M. N. R. A. S.*, **78**, 28. Reply by Eddington, *Ibid.*, **78**, 113 (1917).

[2] *M. N. R. A. S.*, **78**, 36 (1917) and **79**, 319 (1919) "Problems of Cosmogony," Chapter VIII.

[3] *Astrophysical Journal*, **29**, 261 (1909).

[4] *M. N. R. A. S.*, **79**, 2, 1918 and **79**, 177 (1919).

variations in brightness, does not appear to correspond to the phenomena.

The type of pulsation considered by Eddington (which appears far more promising as an explanation of the facts) is a bodily expansion and contraction during which the radius changes by ten or twenty per cent, the temperature rising as the star contracts and falling again as it expands. The changes in temperature account for the variability in light and color, while the outward and inward motions of the surface explain the observed changes in radial velocity; certain important details of the observed variation, especially the wide departure of the oscillations from simple harmonic motion, remain incompletely explained. The conclusion of the most general interest is perhaps that if the energy supply of the star δ Cephei were derived from gravitational contraction alone, its period should be shortening several hundred times as fast as the actually observed rate of change.

6. The problem of the configurations of equilibrium of a rotating incompressible mass of fluid is an old one. Successive stages in its development are marked by the spheroid of Maclaurin, the ellipsoid of Jacobi, and the pear-shaped figure of Poincaré. The stability of this latter figure had been investigated by G. H. Darwin and Liapounoff, with contradictory results, and the problem was only resolved in 1916 by Jeans,[1] who showed that an expansion to the third order of small quantities was required to arrive at decisive results, and that the pear-shaped figure was unstable.

It follows that a slowly contracting and rotating mass of incompressible fluid, after following the series of spheroids and ellipsoids, would become physically unstable, and go through a period of rapid change, and probably break up into two separate masses. The more difficult question of the behavior of a rotating compressible mass is of greater astrophysical importance. This too has been attacked by Jeans.[2] He finds that there are two possible mechanisms of breaking up. The mass may divide into two (or perhaps more) isolated parts by fission, or the centrifugal force at some point or line upon its surface may become equal to the gravitational attraction, and a stream or sheet of matter may be thrown off into

[1] "On the Potential of Ellipsoidal Bodies," *Philosophical Transaction A*, **215**, 27 (1914). "On the Instability of the Pear-Shaped Figure, "*Phil. Trans. A*., **217**, 1 (1916). "Problems of Cosmogony," Chapters IV, V.

[2] "The Configurations of Rotating Compressible Masses," *PhilosophicalTransactions*, A. **218**, 157 (1917). "Problems of Cosmogony," Chapter VII.

space. When the mass is of uniform density the first of these processes happens: when it is greatly concentrated at the center, the second.

Intermediate situations may be represented by assuming that the compressible material obeys the equation of state $p = K\rho^\gamma$ (where $\gamma = \infty$ gives homogeneity while $\gamma = {}^6/_5$ is found to lead to an infinitely great central condensation). It is found that fission takes place if γ exceeds 2.2, while if γ is less than this limit, the spheroidal surface is distorted, develops a sharp edge, and matter is thrown off in a sheet in the equatorial plane. This limiting value of γ corresponds to a surprisingly low degree of central condensation in a spherical mass—the central density being only about three times the mean density.

7. The origin and evolution of binary stars has been considerably discussed. Direct observational evidence, showing pairs of stars revolving almost in contact, makes it almost certain that such systems have been formed by the fission of a single rotating mass, and this explanation may be accepted for the spectroscopic binaries of short period. The densities of these stars are, however, so low that it is doubtful whether the value of γ of the gas of which they are formed can be as great as 2, but Jeans has shown[1] that the influences of ionization and radiation pressure may remove this difficulty.

The wider visual binaries, with periods from five years to many centuries, present a more difficult problem. Moulton[2] and others have proved that their periods can never have been very much shorter than at present, and that, if they were formed by fission, the density at the time of separation must have been exceedingly small. Moreover, their orbits are often very eccentric, and Nölke[3] has shown that tidal friction is incompetent to produce such high eccentricities from the nearly circular initial orbits. Triple and multiple systems (as Russell[4] has shown) exhibit a grouping into close pairs, with widely distant companions (cf. 2 above) which is a necessary consequence of the theory that they have been formed by repeated fission, but may also be a consequence of other modes of origin.

Jeans[5] has pointed out that the effect of encounters between bi-

[1] *Phil. Trans. A.*, **218**, 208 (1917).

[2] F. R. Moulton, *Astrophysical Journal*, **29**, 12–13, 1909.

[3] Fr. Nölke, *Abh. Nat. Ver. Bremen*, **20**, *Teil.* 2, 1911.

[4] H. N. Russell, *Astrophysical Journal*, **31**, 185 (1910).

[5] *M. N. R. A. S.*, **79**, 100 (1918), and **79**, 408 (1919); "Problems of Cosmogony," Chapter XI.

nary systems and other stars which happen to pass near them tend in the long run, to an equipartition of energy between the radial and transverse components of motion, and to an average orbital eccentricity of 0.64—a little greater than the observed mean value. He supposes that most of this action has taken place in the remote past, when the stars were closer together than now (see below) and that the majority of visual binaries probably started as neighboring nuclei when the stars were originally formed (in agreement with a previous conclusion of Moulton[1]).

8. The theory of rotating masses has been employed by Jeans[2] in a bold and brilliant hypothesis of the origin of spiral nebulae, and even of our universe of stars. A huge rotating mass of rarefied gas (or cloud of dust), would, upon contraction, assume a spheroidal form, then as this grew more flattened, become lens-shaped till, in time, matter was thrown off by centrifugal force in the equatorial plane. The tidal forces due to the attraction of the rest of the universe would localize the regions of ejection near two opposite points on the equator, and the gas, streaming out from these, would form two spiral arms enclosing the nucleus.

This is a remarkably good model of a spiral nebula. Moreover Jeans shows that, if the rate of ejection is rapid enough, the outgoing stream of gas will become unstable, and tend to break up into condensations or nuclei under its own gravitational attraction. Condensations of this sort are conspicuous on the photographs of many spiral nebulae. If the nebula is big enough, they may be so massive that they ultimately condense into stars.

The masses calculated for spiral nebulae on this hypothesis are very great—5000 to 500,000,000 times the mass of the Sun—but the mean density is excessively low (4×10^{-17} grams per cubic centimetre). These values are roughly of the order of magnitude indicated by other observational data.

More tentatively, Jeans postulates that a similar huge nebula, in the course of hundreds of millions of years, may have completely dispersed its substance into star-forming condensations, which, circulating in the general plane of the nebula, and spreading out to some sixty times its original diameter may have given rise to the existing galactic system of stars.

[1] See note 2, p. 15.

[2] *M. N. R. A. S.*, **77**, 186 (1917); "Problems of Cosmogony," Chapter IX.

PART III. THE THEORY OF THE PROBLEM OF THREE OR MORE BODIES[1]

When three or more bodies (taken as particles) move according to the Newtonian law of gravitation, the mathematical determination of their motion presents great difficulty. The so-called restricted problem of three bodies in the plane is that special case in which two of the bodies move in circles about their center of gravity, while the third body is of negligible mass and moves in their plane attracted by them. We turn first to this case, which has particular importance for the reason that many of the fundamental characteristics of the more general problem appear in simple form.

THE RESTRICTED PROBLEM OF THREE BODIES IN THE PLANE

(a) *Periodic orbits.*

Hill[2] was the first to realize the importance of certain periodic orbits in the restricted problem of three bodies for the lunar theory. Later G. H. Darwin[3] undertook extensive calculations based on mechanical quadrature and found other classes of such orbits not obtainable by Hill's methods. Moulton and his students[4] have applied the method of analytic continuation of Poincaré to the treatment of various periodic orbits. Brown,[5] Strömgren[6] and others have concerned themselves with types of periodic orbits of particular astronomical significance. In general only a beginning has been made with the determination of all the types of periodic orbits. Rigorously proven qualitative results are rare.[7]

It should be noted that the periodic orbits referred to, form closed curves in the plane rotating with the finite bodies. Orbits of ejection in which the small body collides periodically with one of the finite bodies are included.[4] The singularity of collision can be completely disposed of by mathematical transformation.[8]

As in most dynamical problems there are many types of non-

[1] For a report on recent literature see E. O. Lovett, *Quarterly Journ. Math.*, **42**, 252–315 (1911).

[2] G. W. Hill, *Am. Journ. Math.*, **1**, 5–26, 129–147, 245–260 (1878).

[3] G. H. Darwin, *Acta Math.*, **21**, 99–242 (1897).

[4] F. R. Moulton, *Proc. Math. Cong.*, Cambridge, England, **2**, 182–187 (1913); Also see *Periodic Orbits*, Carnegie Inst., Washington, 1920.

[5] E. W. Brown, *M. N. R. A. S.*, 1911.

[6] E. Strömgren, *A. N.*, **168**, 105–108 (1905); **174**, 33–46 (1907).

[7] G. D. Birkhoff, *Rend. Circ. Mat. Palermo.*, **39**, 265–337 (1915). Other references are given in this paper.

[8] T. N. Thiele, *A. N.*, **138**, 1–10 (1895); T. Levi-Civita, *Acta Math.*, **30**, 305–327 (1906).

periodic orbits. Poincaré was the first to recognize sufficiently the central importance of the periodic orbits and conjectured that every orbit may be approximated to for an arbitrary length of time by a periodic orbit.[1] This conjecture has neither been proved nor disproved, but it has been shown that for very general classes of dynamical problems, including the restricted problem of three bodies, every stable orbit has either certain properties of recurrence, or asymptotically approaches and recedes from orbits with such properties.[2]

(b) *Integrability.*

The concept of integrability is one which admits of various interpretations. Thus, if the differential equations of the problem under consideration are such that the coördinates may be expressed in terms of "known functions" of the time, the dynamical problem may be called integrable. Unfortunately a function which is not regarded as known at one time may be admitted later to the class of known functions. For instance, Painlevé in his Stockholm Lectures[3] admits all functions defined by infinite series which converge uniformly. If this be accepted, it follows immediately that the restricted problem of three bodies is integrable. For, as was stated above, it is possible by change of variables to eliminate entirely the singularities of collision, and then find series of the type required. Unfortunately this type of integrability is of doubtful importance.

Sundman[4] in his epoch-making work on the general problem of three bodies established that this problem also is integrable in the sense of Painlevé. But he was not able to draw any conclusions therefrom, and his series were useless for purposes of computation.

With so many more or less justifiable concepts of integrability, the question arises whether any one is to be preferred before the others. The answer seems to be that the differential equations of a dynamical problem should be called integrable in the vicinity of a particular periodic solution if the formal trigonometric series for this solution and nearby solutions converge for all values real or complex of the variables involved. In this sense it has not yet been demonstrated that the restricted problem of three bodies is not

[1] H. Poincaré, *Les méthodes nouvelles de la Mécanique Céleste*, Paris, 1892–1899.

[2] G. D. Birkhoff, *Bull. Soc. Math. France*, **40**, 305–323 (1912); *Acta Math.*, **43**, 1–119 (1920).

[3] Painlevé, *Lecons sur la Théorie Analytique des Équations Différentielles, professées à Stockholm*, Paris, 1897.

[4] C. F. Sundman, *Acta Math.*, **36**, 105–192 (1912).

integrable although Poincaré has shown that these series do not converge uniformly for all values of one of the parameters of the problem.[1]

(c) *Reducibility.*

In order to obtain a comprehension of the restricted problem of three bodies, it is necessary to deal with the *totality* of orbits for a given value of the energy constant. A partial explanation is the following: For periodic orbits the motion may be followed indefinitely. But a non-periodic orbit may have such complexity as to approach and recede from periodic orbits and more generally to be so related to other orbits that it is not possible to isolate the orbit completely.

Now there are essentially three arbitrary constants involved in the restricted problem of three bodies, namely the two relative coördinates of the particle and the angular coördinate giving the direction of motion. If these three constants be interpreted as rectangular coördinates in space, the totality of orbits may be represented as the stream lines of an incompressible fluid in steady motion in this space. To a periodic motion will correspond a closed stream line. Hence if we imagine the closed stream line to be cut by a stationary surface S, there will correspond to the successive intersections of this surface by a stream line a sequence of points of the surface. The transformation T of the surface which takes each point into the next following one on the same stream line, and in particular takes the point of intersection of the closed stream line with the surface into itself, has a very intimate connection with the dynamical problem. In fact, by this means, first introduced by Poincaré,[1] we are enabled to reduce the restricted problem to the transformation of a surface into itself.

Another way of seeing this reducibility is the following: Consider the direct variational periodic orbit of Hill in the restricted problem of three bodies. Orbits (for the same energy constant) which cross the line of apsides with nearly the same direction and abscissa as this periodic orbit are determined by the abscissa r of crossing and the direction given by an angular variable σ. When the small body projected in this manner crosses a second time after a complete circuit we have a new pair of variables r', σ'. Now r, σ may be regarded as the rectangular coördinates of a point in the plane. That transformation of the plane which carries r, σ into r', σ' consti-

[1] See note 1, p. 18.

tutes a transformation T. It is to be observed that the variational orbit corresponds to an invariant point of the transformation.

It will be found that the important qualitative properties of the orbit are mirrored in corresponding properties of the transformation T. Thus if the problem were integrable in the specific sense referred to above, the transformation near the invariant point would be *exactly* of the nature of a rotation in which the angle of rotation varies with the distance from the invariant point.

As Poincaré showed, not only the orbits near a particular periodic orbit but the totality of orbits can be treated by means of a transformation T. The transformation may be regarded as a transformation of the surface of a sphere into itself, approximately like a rotation in which the angle of rotation varies with latitude. The two fixed points correspond to the fundamental direct and retrograde periodic orbits.[1,2]

(d) *Stability.*

The outstanding problem of stability for the fundamental direct periodic orbit is this: will all slightly disturbed orbits remain indefinitely in the vicinity of this periodic orbit? In terms of the transformation T of a surface S, such stability would imply the existence of an infinite set of invariant closed curves as near as desired to the invariant point corresponding to this periodic orbit. In the interpretation by means of fluid motion, such stability would therefore mean the presence of infinitely many torus-shaped canals of stream lines enclosing the closed stream line which corresponds to the periodic orbit.

Levi-Civita[3] has shown that when the periodic orbit is of such a type that the mean motion of the small body is commensurable with the mean motion of the two other bodies, there will not be stability in this sense. In fact, there will then be orbits approaching and receding from the given periodic orbit. However, there remains the possibility that the degree of instability is limited.

In the case of incommensurable mean motions the formal series are available, and this implies stability in the usual astronomical sense. However, there is a distinction between the type of mathematical stability referred to above and astronomical stability. The astronomical type can be treated by direct computational

[1] See note 1, p. 18.

[2] G. D. Birkhoff, *Rend. Circ. Mat.*, Palermo, 39, 265–337 (1915). Other references are given in this paper.

[3] T. Levi-Civita, *Ann. di Mat. ser.* 3, 5, 221–309. 1901.

methods, but the mathematical type presents the utmost difficulty.

From the practical standpoint one may ask the following two interesting questions. Suppose that the body of small mass is subject to arbitrary slight disturbance, although initially in the fundamental direct periodic orbit. What is the least time in which the particle can deviate by a stated amount from that orbit? What is the probable length of time that will elapse before it deviates by this amount? Neither of these questions appear to have received consideration despite their interest for the lunar theory.

THE PROBLEM OF THREE OR MORE BODIES

Most of the facts outlined above have their analogues in the more general problem.

The great increase in complexity precludes any attempt, at an enumeration of the types of periodic orbits. Nevertheless such orbits must be basic in any attempted treatment of the questions which arise.

The question of ideal collision has been partially disposed of in the fundamental paper of Sundman referred to above and in later papers of Levi-Civita.[1] Sundman established that when the three bodies do not move in one plane a triple collision is impossible and that at double collision the behavior of the colliding bodies is essentially the same as in the ordinary two-body problem.

Sundman went further and proved that the sum of the three mutual distances will exceed always a specified positive constant.

Birkhoff has announced that the work of Sundman can be extended to apply not only to the usual problem of three bodies attracting each other under the Newtonian law, but to n bodies under similar laws, and that furthermore Sundman's methods may be applied to give the conclusion that if the area integral constants be assigned and if the mutual distances are small enough initially, the sum of the mutual distances increases indefinitely with lapse of time.

For periodic motions the sum of these distances remains finite of course, and the same is true of motions asymptotic to these periodic motions. It appears possible, however, that in all other cases the sum of the three distances increases indefinitely.

Thus in an idealized earth, sun, moon problem it is likely that

[1] T. Levi-Civita, *Acta Math.*, **42**, 99–144 (1919).

the earth and moon will recede from the sun while the moon approaches the earth.

In the case of n nearby bodies we may anticipate that one or more will recede gradually from the others with lapse of time until finally the bodies are all remote from each other or moving in nearby pairs as in the two body problem. The source of the potential energy required will of course lie either in the high initial velocities or in the near approach of the paired bodies. From time to time, then, these bodies may approach so near as to collide. These conjectures seem in harmony with the facts of stellar distribution.

The interest of the pure mathematician in the problem of three or more bodies has been stimulated by its importance for an understanding of the past and future of the stellar universe. The entrance upon the field of the theory of relativity of Einstein has altered this situation considerably. If the relativistic point of view prevails there can be little doubt that new factors of the utmost importance will be introduced in astronomical speculation concerning great lapses of time, although for limited intervals of time the classical problem of three or more bodies will maintain its importance. Only the very simplest features of the modifications required by the theory of relativity have as yet been determined, mainly those for a very small body in the presence of a central body.[1]

[1] K. Schwarzschild, *Sits. Preuss. Akad. Wiss.*, **35**, 189–196(1916). See also W. De Sitter, *M. N. R. A. S.*, **76**, (1916).

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SECONDARY RADIATIONS PRODUCED BY X-RAYS

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BULLETIN

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Vol. 4, Part 2 OCTOBER, 1922 Number 20

SECONDARY RADIATIONS PRODUCED BY X-RAYS,

AND SOME OF THEIR APPLICATIONS TO PHYSICAL PROBLEMS[1]

By Arthur H. Compton

Professor of Physics, Washington University

Contents

I. Secondary X-rays

By the term "secondary" X-rays is meant any radiation of the X type excited by the passage through matter of primary X-rays. "Scattered" radiation signifies the radiation emitted by the electrons in matter (that due to the positive nuclei is theoretically negligible in comparison) due to accelerations to which they are directly subjected by the primary rays. "Fluorescent" radiation is radiation of the energy absorbed from the primary beam and stored temporarily in the kinetic and potential energies of the electrons. In certain cases, as will be seen, it is difficult to distinguish experimentally between these two types of secondary radiation.

[1] This monograph is the third and last of a series which forms the report of the Committee on X-Ray Spectra, of the Division of Physical Sciences, of the National Research Council. This committee consists of the following members: William Duane, Harvard University, Chairman; Bergen Davis, Columbia University; A. W. Hull, General Electric Research Laboratory; D. L. Webster, Leland Stanford Junior University; Arthur H. Compton, Washington University.

Methods of Studying Secondary X-rays.—The usual method of investigating secondary X-rays may be explained by reference to Figure 1. Radiation from the target S of an X-ray tube, or from some other source of X- or γ-rays, is allowed to traverse a radiator R. This radiator is then found to emit X-rays in all directions. These rays may be investigated by means of an ionization chamber I which is carefully screened from the primary beam.

If the radiator consists of a plate of matter so thin that the X-rays are not appreciably diminished in intensity on traversing it, the intensity I_s of the secondary beam as it enters the ionization chamber may be written as

$$I_s = r_\theta \cdot I\, V/l_2^2,$$

where I is the intensity of the primary beam at R, V is the volume of the radiator, l_2 is the distance from the radiator to the ionization cham-

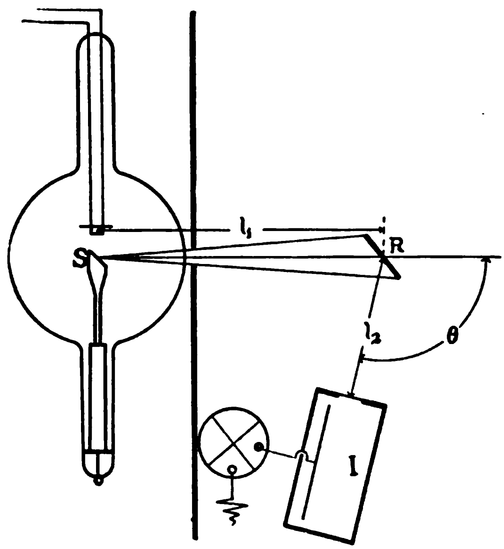

FIGURE 1. X-rays from the target S of the X-ray tube excite in the radiator R secondary rays which are measured by the ionization chamber I.

ber, and r_θ is a constant of proportionality which may be called the "radiating coefficient for the angle θ." Experiment shows that this coefficient is a function of the wave-length of the incident rays, the composition and physical state of the radiator, and the angle θ.

In order to learn the physical significance of the secondary radiation, it is important to know what part of it is scattered and what part is fluorescent. Two methods of making this distinction have been employed. The first depends upon Stokes' law, according to which the wave-length of the fluorescent radiation should be greater than that of the primary radiation which excites it. While this law has not been found to hold universally in optics, no indication of its failure has appeared in the X-ray region. The truly scattered rays, however, are presumably of the same wave-length as the primary rays (cf. infra, p. 18). Thus, since the longer wave-lengths of X-rays are the more strongly absorbed, it is possible by a comparative study of the absorption of the primary and the secondary rays to determine what part of the secondary beam is identical in character with the primary beam. This part constitutes the truly scattered rays, while the remaining less penetrating part of the secondary rays is according to this criterion fluorescent.

A more direct and certain method of separation consists in comparing the spectrum of the secondary with that of the primary X-rays. If the primary ray is homogeneous, and if the electrons traversed are at rest, so that no Doppler effect occurs, the scattered beam will be homogeneous and of the same wave-length; whereas the fluorescent rays will differ in wave-length from the primary. This method, while affording a very definite means of testing for the existence of the two types of secondary rays, is not readily adaptable to quantitative measurement of their relative intensities.

a. The Scattering of X-rays

By electrons acting as point charges.—It is a direct consequence of the electromagnetic theory of X-rays that scattering should occur. For every electron in matter traversed by primary X-rays will be subject to accelerations by the electric vector of the rays, and in virtue of its acceleration will itself radiate energy. On the basis of the assumptions:

1. That the classical electrodynamics is applicable,

2. That the forces of constraint on each electron in matter are negligible,

3. That the electrons scatter independently of each other, and

4. That the size of the electron is negligible, J. J. Thomson showed[1] that the intensity of the X-rays scattered per unit volume of matter traversed by the X-rays, to a point at distance L and at an angle θ with the primary beam, is

$$I_s = N_n I_0 = I \frac{N_n \; e^4(1+\cos^2\theta)}{2m^2L^2c^4}, \qquad (1)$$

where N is the number of negative electrons in the atom, n the number of atoms per unit volume, I_0 is the intensity of the scattered beam due to each electron, I the intensity of the primary beam, e and m the charge and mass of the electron, and c the velocity of light. Using moderately soft X-rays and carbon for the radiating material, Barkla and Ayers[2]

[1] J. J. Thomson, "Conduction of Electricity through Gases," 2nd Ed., pp. 321 et seq.

[2] Barkla and Ayers, Phil. Mag., 21, 275 (1911).

found that the factor $(1+\cos^2\theta)$ accounted satisfactorily for the relative intensity of the secondary rays at angles θ greater than 40°; and later work by Barkla[1] showed that this formula expressed quantitatively the intensity of the secondary rays at 90° from the lighter elements if N is taken as about half the atomic weight.[2] Since the view that the number N is identical with the atomic number has recently been strongly confirmed, this means that under certain conditions equation (1) gives an adequate quantitative expression of the scattering. Under these conditions, of moderately soft X-rays and scattering material of low atomic weight, the assumptions employed by Thomson therefore seem to be justified.[3]

Scattering by Groups of Electrons.—Using soft X-rays and scattering material of higher atomic weight, Owen,[4] Crowther,[5] Barkla and Dunlop[6] and others have found that the rays scattered in the forward direction are more intense than those scattered backward. This phenomenon of "excess scattering" is illustrated in Figure 2, in which the squares represent the scattering of soft X-rays by filter paper as

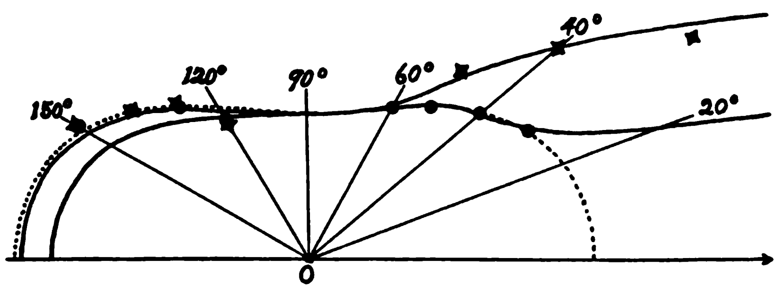

FIGURE 2 Scattering of Soft X-rays (★) and moderately hard X-rays (●) by filter paper, according to Owen. The dotted curve represents theoretical values for a random grouping, the solid curves for an empirical arrangement of the electrons in the atoms.

observed by Owen, and the dotted line shows the relative scattering at different angles calculated from Thomson's relation. The excess of the scattering over the theoretical value increases with the atomic number of the scattering material and with the wave-length of the X-rays employed. This is clearly shown by certain experiments due to Barkla and Dunlop,[6] shown in figure 3. Here the ratio of the intensity of the rays scattered at 90° per unit mass by copper, silver, tin and lead re-

[1] C. G. Barkla, Phil. Mag. 21, 648 (1911).
[2] Recent work by A. H. Compton, indicates that part of the secondary rays studied by Barkla and his collaborators may have been fluorescent in nature; but Compton's experiments lead to the same value for N as those of Barkla.
[3] A. H. Compton, Phys. Rev. 18, 96 (1921).
[4] Owen, Proc. Camb. Phil. Soc., 16, 165 (1911).
[5] Crowther, Proc. Camb. Phil. Soc. 16, 367 (1911); Proc. Roy. Soc., 86, 478 (1912).
[6] Barkla and Dunlop, Phil. Mag. 31, 229 (1916).

spectively to that of the rays scattered by aluminium is plotted against the effective wave-length. Expression (1) indicates that the relative mass scattering by the different elements should be proportional to the number of electrons per gram, that is, that I/I_{Al} should be nearly unity in all cases. It appears from these experiments that this relation may be exact for sufficiently short wave-lengths, but does not hold for the heavier elements at moderate wave-lengths.

The quantitative support of Thomson's theory in the special cases first considered, gives confidence in the application of the classical electrodynamics to the problem of scattering. Any effect due to the forces

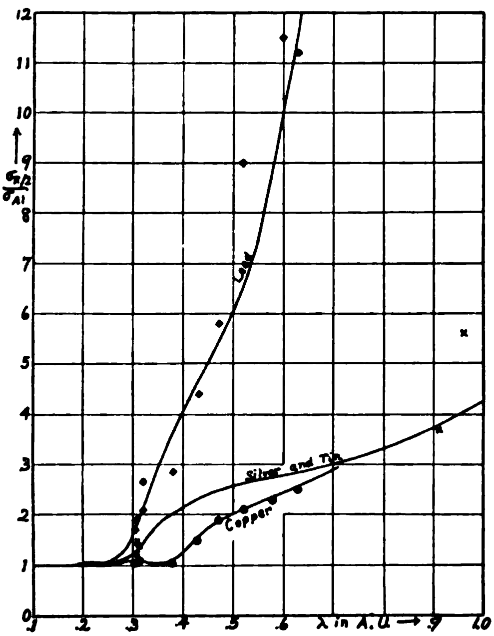

FIGURE 3. Relative scattering of X-rays at 90° per gram of various metals compared with that by aluminium. Data, Barkla and Dunlop; curves, theory based on empirical distribution of the electrons in the atoms.

of constraint on the scattering electrons would be to modify the amplitude of their motion, thus changing the intensity equally in all directions, and therefore could not account for the asymmetrical scattering in the forward and backward directions. If the size of the electron were appreciable, the difference in phase of the rays scattered by its different parts would result in a reduced instead of an increased intensity such as is observed. Thus in order to account for the excess scattering of soft X-rays by heavy elements, Thomson's original assumption that the electrons in matter act independently of each other must be modified.

The suggestion that the electrons in the heavy elements co-operate in their scattering seems to have been made first by Webster,[1] and was first stated in a satisfactory qualitative form by Darwin.[2] Debye[3] and Thomson[4] have solved independently the problem of the scattering of X-rays by atoms (or groups of atoms) consisting of electrons arranged at fixed distances from each other, taking into account the phases of the rays scattered by the different electrons. Their result may be put in the form

$$I_\theta = I_1 \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{\sin\left(\frac{4\pi S_{mn}}{\lambda}\sin\frac{\theta}{2}\right)}{\frac{4\pi S_{mn}}{\lambda}\sin\frac{\theta}{2}}, \tag{2a}$$

where I_θ is the intensity of the ray scattered at an angle θ by the group of electrons, I_1 is the intensity due to a single electron (according to Debye and Thomson identical with the I_θ of equation 1), N is the number of electrons in the group, and S_{mn} is the distance from the mth to the nth electron.

The more general problem of the scattering by atoms composed of electrons in relative motion was investigated by Schott[5] with unsatisfactory results.[6] Glocker and Kaupp,[7] however, have recently calculated the scattering by atoms composed of two or three coplanar rings of electrons revolving at different speeds. Glocker[8] has also calculated the scattering to be expected from Lande's pulsating tetrahedronal carbon atom, and finds a result practically the same as that for Bohr's plane carbon atom. This confirms the conclusion which had been reached by the writer,[9] that the scattering by groups of electrons in the atom depends principally upon the distances of the electrons from the enter, and only slightly upon their spatial distribution. The approximation is thus justified of calculating the scattering on the assumption that the electrons are arranged at random on the surface of shells of radii ρ_s. On this basis the intensity of the beam scattered by an atom is,[9]

$$I_\theta = I_1\left\{N + 2\sum_{1}^{N/2}\left(\frac{\sin 2k_s}{2k_s} - 2\frac{\sin^2 k_s}{k_s^2}\right) + 4\left(\sum_{1}^{N/2}\frac{\sin k_s}{k_s}\right)^2\right\}, \tag{2b}$$

[1] D. L. Webster, Phil. Mag. 25, 234 (1913).
[2] C. G. Darwin, Phil. Mag. 27, 325 (1914).
[3] P. Debye, Ann. d. Phys. 46, 809 (1915).
[4] J. J. Thomson, manuscript read before the Royal Institution in 1916, and loaned to the writer.
[5] G. A. Schott, Proc. Roy. Soc. 96, 695 (1920).
[6] Cf. A. H. Compton, Washington University Studies, 8, 98 (1921).
[7] Glocker and Kaupp, Ann. d. Phys., 64, 541 (1921).
[8] R. Glocker, Zeitschr. f. Phys., 5, 54, (May 10, 1921).
[9] A. H. Compton, Washington U. Stud., 8, 99 (January, 1921).

where N is the atomic number and

$$k_s = \frac{4\pi\rho_s}{\lambda}\sin\frac{\theta}{2}.$$

This formula is much simpler in its application than those of Debye and Glocker, and it leads to equally reliable information concerning the distances of the electrons from the centers of the atoms. If sufficiently refined measurements of the scattering can be made, however, it may be possible to distinguish between the spatial arrangements considered in the different formulae.

The solid lines of Figures 2 and 3 are calculated* from formula (2b), assuming particular arrangements of the electrons in the atoms concerned.[1] These figures show that by taking into account the phase relations between rays from the various electrons in the atoms, a satisfactory explanation may be given of the excess scattering of soft X-rays by heavy atoms.

Scattering by electrons of appreciable dimensions.—When X-rays of very short wave-length are scattered by the lighter elements, the intensity of the scattered beam is less than that demanded by expression (1). For example, Ishino has found[2] that the total secondary gamma radiation averaged over different angles is less than one fourth of the energy calculated by expression (1) for the scattered radiation. Barkla and White pointed out[3] that similar reduced scattering occurs when sufficiently hard X-rays are used, when they observed a total absorption in paraffin less than the energy which should be lost according to Thomson's theory due to scattering alone. These results are supported and extended by A. H. Compton's measurements[4,5] of the part of the

[1] In figure 2, the squares represent measurements with an equivalent spark gap of 2.5 cm., which has been taken to mean a uniform distribution of energy over wave-lengths between 0.5 and 1.0 Å. U. The circles correspond to a spark gap of 7 cm. (λ from 0.25 to 0.50 Å. U.). In figure 3, the wave-lengths estimated by the experimenters were used in the calculations. The radius of the electron was taken to be 2.6×10^{-10} cm., as described below, but this size is almost negligible for the wave-lengths considered. The numbers of electrons at different distances from their atomic centers as employed in the calculations are as follows:

Hydrogen	Oxygen		Carbon		Aluminium		Copper		Silver & Tin		Lead	
No.	No.	Dist. (A.U.)	No.	Dist.	No.	Dist.	No.	Dist.	No.	Dist.	No.	Dist
1	2	.26	2	.35	2	.12	2	.052	2	.036	2	.022
	6	.42	4	.6	8	.26	10	.104	10	.073	10	.045
					3	.7	8	.24	8	.17	16	.090
							8	.42	16	.34	16	.132
							1	1.05	8	.51	16	.202
									4½	.7	16	.31
											6	.6

While these exact distributions are of little value, because of the unreliability of the scattering measurements on which they are based, yet as to order of magnitude the results can hardly be wrong, and they represent the most direct experimental determinations of these distances that have so far been made.

[2] M. Ishino, Phil. Mag., 33, 129 (1917).

[3] Barkla & White, Phil. Mag., 34, 275 (1917).

[4] A. H. Compton, Phil. Mag., 41, 749 (1921).

[5] A. H. Compton, Phys. Rev., 18, 96 (1921).

* See Note 6, page 6.

secondary radiation which has the same wave-length as the primary X-rays (cf. infra p. 16). On the view that these rays only are truly scattered, his data, shown by the solid lines in Figure 4 for the case of paraffin, indicate that at small angles with the primary beam the scattering of hard X-rays by light elements is approximately that given by Thomson's expression (1). At larger angles, however, the intensity of the scattered beam is decidedly less than the theoretical value, the difference increasing for shorter wave-lengths. Indeed, for hard gamma rays he finds the "truly scattered" rays at large angles less than .001 of Thomson's theoretical value.[1]

There appears to be no possibility of accounting for this reduced scattering on the basis either of any grouping of electrons within the

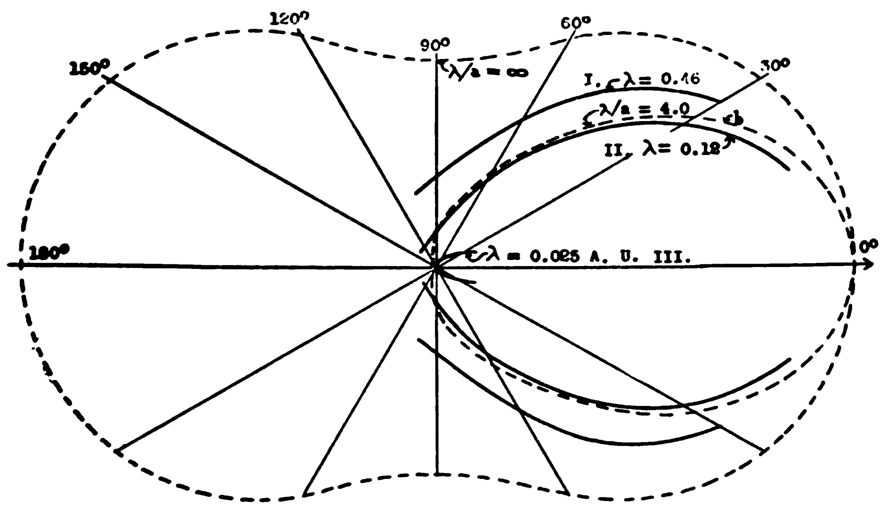

FIGURE 4. "True" scattering of hard X-rays and gamma rays by paraffine. Solid lines, experiment (Compton); broken lines theory for electron of radius *a*.

atoms or of any forces of constraint upon the electrons.[2] Unless the classical electrodynamics ceases to be applicable to these very high frequency rays, it therefore seems necessary to modify Thomson's assumption that the electron acts as a simple point charge of electricity.[3] It is obvious that if the electron is comparable in diameter with the wave-length of the X-rays, partial interference will occur between the rays scattered from its different parts, reducing the intensity of the

[1] For the data on γ-rays, cf. note 4, p. 7 and A.H.Compton, Phil. Mag. 41, 770 (1921). In figure 4 the *absolute* values of the scattering are compared with Thomson's formula, assuming that the number of effective electrons is equal to the atomic number. Figure 2 shows merely the *relative* scattering at different angles.

[2] A. H. Compton, Phys. Rev. XIV. p. 31 (1919); Washington U. Studies, 8, 96 (1921).

[3] For a possible modification of electrodynamics which may account for this reduced scattering as well as certain other phenomena, cf. infra, pp. 18 and 19.

scattered beam, and that this interference will be more nearly complete for rays scattered at the larger angles.

The theory of the scattering of high frequency radiation by matter containing electrons of appreciable size, taking into account this interference between the rays scattered by the different parts of the same electron, has been discussed in a series of papers by A. H. Compton[1] and also by G. A. Schott.[2] The magnitude of the scattering in this case is a function of the form of the electron, its rigidity, and the ratio of the charge to the effective mass of its different parts, as well as the wave-length of the X-rays. In the case of a spherical shell electron of negligible rigidity, the intensity of the ray scattered at an angle θ by a single electron is found to be*

$$I_1=\frac{I_0 \sin^2 x}{x^2} \qquad \left[x=\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right], \tag{3a}$$

where I_0 is given by expression (1), a is the radius of the electron, and λ is the wave-length of the X-rays. Similarly if the electron is in the form of a ring of negligible rigidity, the scattering is* †

$$I_1=I_0\cdot\frac{1}{x}\sum_0^\infty J_{2n+1}(2x), \tag{3b}$$

where J_s is Bessel's J function of the sth order, and the other quantities have the same significance as before. A solid sphere, without rigidity, with volume density of electric charge inversely proportional to the square of the distance from the center, and of equal ratio e/m of charge to mass everywhere, should scatter according to the expression,*

$$I_1=I_0\left\{1-\frac{x^2}{3\cdot 3!}+\frac{x^4}{5\cdot 5!}-\cdots\right\}^2. \tag{3c}$$

The scattering by rigid electrons and by electrons whose parts have different ratios of charge to mass has not been examined mathematically. It can be shown, however, that if the electron possesses rigidity, the scattering is not a function of x alone, but is a more complicated function of λ and θ.

In Figure 5, curves a, b and c show respectively the values of I_1 as calculated according to expressions 3a, 3b, and 3c, plotted as functions of a/x. That these formulae describe satisfactorily the relative scattering at different angles is shown by the dotted curve b of Figure 4, which is calculated for $\lambda/a=4.0$ from expression (3a). But when the experimental data represented in Figure 4 by the solid lines, I, II, and III

[1] A. H. Compton, Journ. Wash. Acad. Sci., Jan. 1, 1918; Phys. Rev. 14, 20 (1919); Washington U. Stud. 8, 93 (1921).

[2] G. A. Schott, Proc. Roy. Soc. 96, 695 (1920).

* See Note 1, page 9. † See Note 2, page 9.

are transferred to Figure 5 (represented in this figure by broken lines), we see that the agreement is not perfect. Though the general variation of I_1/I_0 with λ and θ is in accord with the calculated values, it seems that the relative scattering cannot be described accurately as a function of x. This is shown by the fact that if, as in the figure, curve II for $\lambda = .12$Å coincides with the theoretical curves, curve I for $\lambda = .45$Å departs rather widely from the theory. According to the results of the

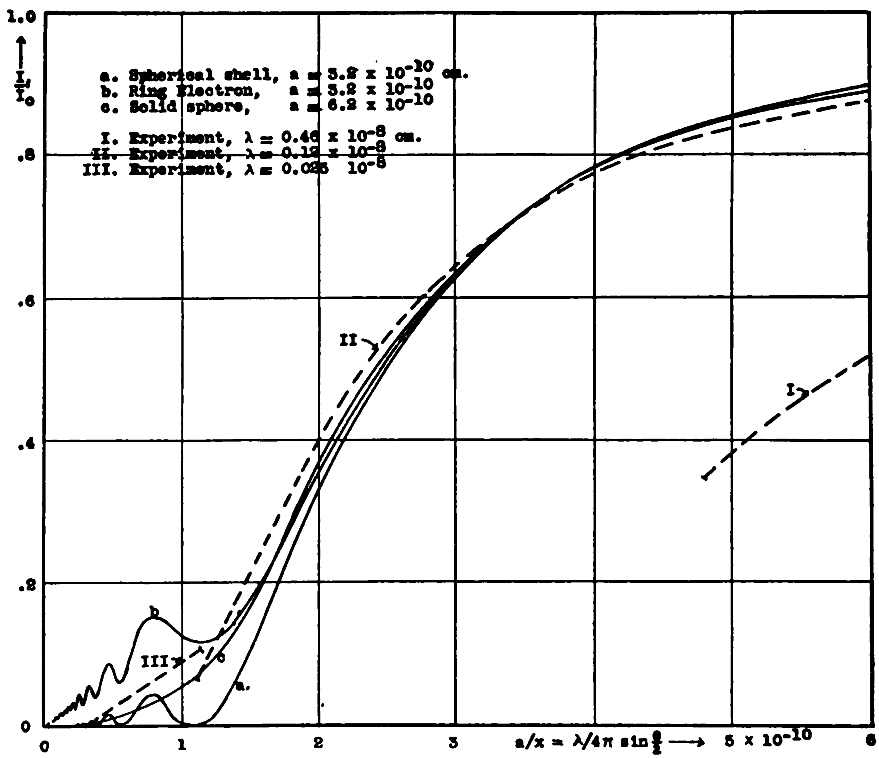

FIGURE 5. Scattering of hard X-rays, showing the departure of the experiments (broken lines) from $I^1/I_0 = 1$, and the approximate agreement with the theory (solid lines) based on the view that the electron's radius is comparable with the wave-length.

last paragraph, the fact that I_1/I_0 is not a function of x only would indicate that the electron has appreciable rigidity. It is doubtful, however, if the assumption of rigidity in the electron would modify the theoretical scattering in the manner demanded by the experiments. This is an important matter for further theoretical and experimental investigation, for if this discrepancy is real it would not seem possible to reconcile the results with the classical electrodynamics.

The important thing, however, is that if the electrons had dimensions comparable with 10^{-13} cm., as usually assumed, the classical theory requires scattering represented by the upper line of Figure 5, where $I_1/I_0=1.0$. On this basis, the fact that experiment gives consistently lower values when short wave-lengths are used, indicates that the electron is not sensibly a point charge of electricity. We find, indeed, that this reduced scattering for small values of a/x can be accounted for by interference between the rays from different parts of the electron, if its radius is of the order of 4×10^{-10} cm.

Scattering by Small Crystals.—An important special case of the scattering by groups of electrons is that of the scattering by a mass of minute crystals with random orientation. This includes, for example, the scattering by the ordinary inorganic solids. Debye and Scherrer[1] first pointed out that in such a random assemblage of crystals some will be so oriented as to reflect X-rays from every plane which can be drawn parallel to layers of atoms in the crystal. Thus a beam of X-rays passing through such a mass of crystals is partially scattered in a series of coaxial cones, the semi-apex angle θ of each cone being determined by the relation

$$n\lambda=2D\sin\frac{\theta}{2},$$

where D is the distance between the layers of atoms giving rise to the reflection, and n is the order of reflection. Since every possible layer of atoms thus gives rise to a reflected beam which may be recorded on a photographic plate, these "powdered crystal spectrograms" are of great value as a means of studying crystal structure. Extensive applications of this method have been made by Debye and Scherrer[2] and especially by Hull[3] in determining the arrangement of the atoms in substances which cannot readily be obtained in the form of large crystals.

If motions of the electrons are neglected, the intensity of the beam scattered by such a mass of minute crystals is given by Debye's formula (2a), when the double summation is performed over all the electrons in each component crystal. This process leads, however, to an expression containing so many terms that it is quite unmanageable. A more satisfactory procedure is to start with the equation for the reflection of X-rays by crystals as given by Darwin[4] and A. H. Compton,[5] which applies if (as is the case experimentally) the crystals are large enough to give a spectrum line whose angular width is small compared with θ. By the application of this method it can be shown[6] that the total energy

[1] Debye and Scherrer, Phys. Zeits. 17, 277 (July 1, 1916).
[2] Debye and Scherrer, Loc. cit. and ibid., v. 18, 291 (1917), v. 19, 23 (1918) et al.
[3] A. W. Hull, paper Amer. Phys. Soc., Oct. 25, 1916; Phys. Rev. 9, 85 (1917), 10, 661 (1917) et al.
[4] C. G. Darwin, Phil. Mag., 27, 325 (1914).
[5] A. H. Compton, Phys. Rev. 9, 29 (1917).
[6] It is hoped to publish this calculation soon in the Phys. Rev.

scattered by a layer of crystals of thickness δx in the cone whose semi-apex angle θ is defined by $n\lambda = 2D \sin\frac{\theta}{2}$ is

$$E_\theta\,\delta x = E_i\cdot\frac{nN^2\lambda^3}{2\sin\theta/2}\,\frac{\rho'}{\rho}\varphi^2P\psi^2D\cdot\delta x, \tag{4}$$

where:

E_i = incident energy of wave-length λ,

n = number of planes in crystal giving rise to reflection at angle θ (e. g. for [100] planes $n=3$, for [110] $n=6$, for [111] $n=4$, etc.)

N = number of electrons per unit volume in each component crystal,

$\frac{\rho'}{\rho}$ = ratio of density of mass of crystals to density of individual crystals,

$\varphi = e^2/mc^2$, $\qquad P = 1+\cos^2\theta$,

ψ = function of the arrangement of the electrons in the atoms,

$$=\int_a^b F(z)\cos\left(\frac{4I\sin\theta/2}{\lambda}\right)dz,$$

where

$b-a$ = diameter of the atom,

$F(z)$ = probability that a given electron will be at a distance z from the mid-plane of the layer of atoms to which it belongs,

D = factor accounting for the thermal agitation of the atoms,

$$= e^{-\beta\frac{\sin^2\theta/2}{\lambda^2}},$$

where β is a constant characteristic of the crystal, which has been evaluated theoretically by Debye[1] (cf. also this report, vol. 1, p. 420).

If, as is usually the case in practice, the thickness δx of the mass of crystals is great enough to produce appreciable absorption, the reduction in intensity of the scattered beam due to this cause must also be taken into account.

For investigations of the arrangement of the atoms in crystals, in which the different angles θ at which scattered rays appear are of first importance, the scattering of X-rays by powdered crystals has been studied only by the photographic method. It has been customary to have the scattering material in the form either of a thin plate or of a narrow cylinder placed perpendicular to the incident beam, between the target and the photographic plate. W. H. Bragg has coated a flat plate with a layer of the powdered crystals, and has used this plate in place of the crystal of an X-ray spectrometer.[2] When the two arms of the spectrometer are equal, X-rays are scattered into the slit of the ionization chamber from all parts of the plate at approximately the angle θ.[3] With

[1] P. Debye, Ann. d. Phys., 43, 49 (1914); A. H. Compton, Phys. Rev. 9, 47 (1917)
[2] W. H. Bragg, Proc. Phys. Soc., 33, 222 (1921).
[3] Cf. e.g. Bragg's "X-rays and Crystal Structure," p. 26.

this arrangement he has succeeded in obtaining sufficient intensity to measure the scattered beam by the ionization method. The theoretical scattered energy in this case is*

$$E_s = E_\theta \cdot \frac{\left(l\,(1 - e^{\frac{-2\mu z}{\sin\theta/2}}\right)}{4\ \pi L\ \mu \sin\theta}, \tag{5}$$

where E_θ is given by equation (4), E_i being the total energy falling on the layer of powdered crystals; L is the distance of the ionization chamber from the plate, l is the vertical height of the slit in the ionization chamber, μ is the absorption coefficient of the X-rays in the crystal mass, and z is the thickness of the layer of crystals. This formula supposes that the slit in the ionization chamber is broad enough to receive all the rays scattered at the angle θ, and that its height is small enough so that the curvature of the image of the scattered beam will not be great.

Expression (5) gives the intensity of the scattered rays in terms of known quantities, except for the function $F(z)$ of the arrangement of the electrons in the atoms.[1] Experimental determinations of the intensity E_s for different angles θ should therefore be of great value in determining the distribution of the electrons, and Bragg's preliminary measurements indicate that the method is feasible. Work of this kind with powdered crystals has a distinct advantage over similar work on the reflection from large crystals in that the absorption coefficient involved in equation (5) is directly measureable, whereas the absorption coefficient of X-rays in a large crystal at the angle of maximum reflection is very difficult to determine. It would seem that studies of the intensity of the scattering by such groups of crystals may afford most valuable information concerning the arrangement of electrons in atoms.

An interesting theoretical problem in this connection which awaits solution is to determine the relation between the scattering by atomic groups of electrons as expressed by equation (2) and the scattering by crystalline groups of electrons, as expressed by equations (4) and (5). It is found that a large part of the scattering of homogeneous X-rays by solid substances occurs in definitely defined cones, as is to be expected if these substances consist of finely divided crystals.[2] But the work of Barkla and others with heterogeneous rays indicates, as we have seen, scattering at all angles in accord with equations (2) based on independent

[1] The temperature factor D is uncertain to some extent, depending upon the question of the existence of a zero-point energy of thermal oscillation. This question, however, may possibly be answered by measurements of the scattering of X-rays by crystals at different temperatures.

[2] Recent experiments by G. E. M. Jauncey (Phys. Rev., 19, 435, 1922), however, show that even for the best crystals much more energy is spent in diffuse scattering than in crystalline reflection.

* See Note 6, page 11.

scattering by the different atoms. Thus it would appear that the energy in the spectrum lines from the fine crystals, when averaged over relatively large angles, must equal the energy scattered by the substance in an amorphous form.

An important difference between the scattering by small crystals and that by independent molecules lies in the fact that there can be no crystalline reflection at an angle smaller than that defined by $\lambda = 2D_{max} \sin\frac{\theta}{2}$, or

$$\theta_{min} = 2 \sin^{-1}(\lambda/2D_{max}),$$

where D_{max} is the largest grating space for any set of planes in the crystal, and λ is the wave-length of the incident radiation. Thus one might have predicted Hewlett's observation[1] that the scattering of X-rays of wave-length about .7 A.U. by powdered diamond and graphite becomes very small[2] at angles less than 5°. For molecules arranged at random, however, expressions (2) indicate very strong scattering at small angles, approaching N times the normal value (equation 1) as θ approaches zero. Hewlett finds,[4] however, even in the case of a liquid (mesitylene, $C_3H_6(CH_3)_3$) that the intensity of the scattered beam approaches zero at small angles. It follows that there must be even within the liquid sufficiently large groups of regularly arranged atoms to produce nearly complete interference at small angles. In order that formulas (2) may be valid at all angles, it would therefore seem possible to apply them only to the case of gases.

b. The Fluorescence of X-Rays

There exist several types of "characteristic," fluorescent radiation, consisting of a number of nearly homogeneous rays characteristic of the substance employed as radiator. There is some evidence also for the existence of a "general" fluorescent radiation, whose character depends principally upon that of the primary rays. These two types of fluorescent rays correspond closely to the characteristic and general radiations respectively from the target of the X-ray tube.

Characteristic Fluorescent X-rays

The characteristic fluorescent X-rays, discovered by Barkla and Sadler,[3] consist of the K, L, etc. series radiations from the substance traversed by the primary rays. These rays, therefore, have properties identical with those of the same radiations when excited at the target of an X-ray tube, which have been described in the other sections of this

[1] C. W. Hewlett, paper before Am. Phys. Soc. Nov. 26, 1921.

[2] Some recent experiments by A. R. Duane and W. Duane show that the energy scattered at an angle less than θ_{min} is about 1 per cent for aluminium and less than 5 per cent for water at an angle a degree or so larger than θ_{min}. They have been using this principle to measure the distances between crystal planes.

[3] Barkla & Sadler, Phil. Mag. 16, 550 (1908).

[4] See Note 1, page 14.

committee's report.[1] The characteristic fluorescent rays, are, however, usually mixed with much less general radiation than is the case with primary rays, and have therefore been much used as a source of comparatively homogeneous rays.

Experiments by Sadler[2] have shown that the K-rays from one element will excite the K-rays from all lighter elements, but not from the same or heavier elements. His results strongly suggest that the K, L_1, L_2 L_3, etc. series radiations characteristic of any particular element are excited only by rays of wave-length shorter than the critical K, L_1, etc. absorption wave-lengths (cf. this report Vol. I, p. 386) characteristic of the element. It appears, however, that no direct quantitative experiments have been performed to test this point.

The experiments of Barkla and his collaborators have shown that the characteristic fluorescent radiation emitted by an atom is unpolarized and is distributed uniformly in all directions. Apparently no quantitative measurements have been made of the intensity of this characteristic radiation. When it does occur, however, it is usually much more intense than the scattered radiation. The energy which goes into the radiation comes from that absorbed by the atom. Since the absorbed energy is closely proportional to λ^3 (cf. infra. p. 36), it is probable, if the incident waves are shorter than the critical wave-length, that the intensity of the characteristic fluorescent radiation is proportional to λ^4.

General Fluorescent X-rays

The softening of Secondary X-rays.—Under this head we shall discuss the fact that, even when no appreciable characteristic fluorescent radiation is excited, the secondary radiation is always somewhat less penetrating than the primary radiation which excites it. Secondary γ-rays are much softer than the primary γ-rays,[3] the backward secondary rays being both less penetrating and less intense than the rays in the forward direction. As a result also of careful experiments on the secondary radiation excited in carbon by the characteristic X-rays from different substances, Sadler and Mesham[4] found that this radiation was less penetrating than the primary beam, and that this difference in quality was greater the harder the primary rays employed.

It has been generally supposed[5] that this softening of the secondary radiation is due to the fact that the softer components of the beam were

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[1] W. Duane, This Bulletin, vol. I, p. 383.

[2] C. A. Sadler, Phil. Mag. 18, 107 (1909).

[3] Eve., Phil. Mag. 8, 669 (1904); R. D. Kleeman, Phil. Mag. 15, 638 (1908); Madsen, Phil. Mag. 17, 423 (1909). D. C. H. Florance, Phil. Mag. 20, 921 (1910), et al.

[4] Sadler and Mesham, Phil. Mag. 24, 138 (1912).

[5] e.g. Florance, loc. cit.; Oba, Phil. Mag. 26, 601 (1914); A. H. Compton, Phys. Rev., 14, 20 (1919); K. W. F. Kohlrausch, Phys. Zeit. 21, 193 (1920), et al.

more strongly scattered than the harder ones. This may account for a part of the effect when γ-rays are employed, but Sadler and Mesham showed that in their experiments it was the harder components which were the more strongly scattered. In the case of γ-rays also, Gray[1] established the fact that only a small part of the softening was due to this cause, the greater part of the effect being due to a real change in

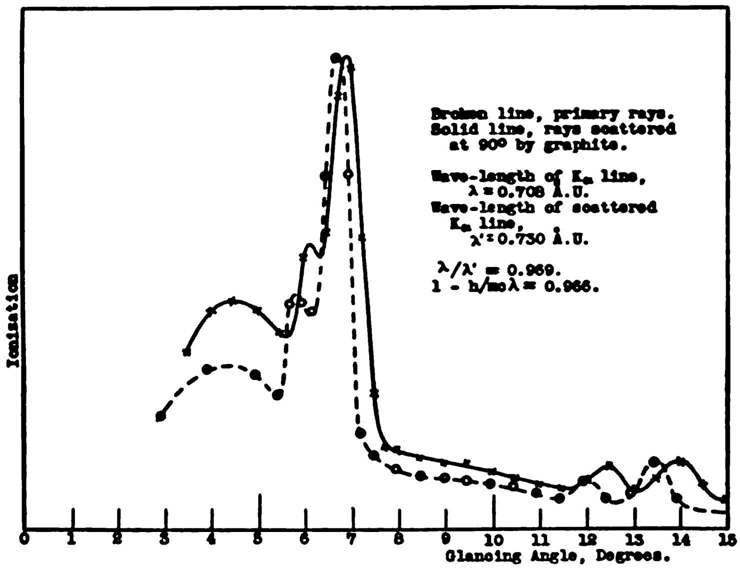

FIGURE 6. Spectrum of scattered X-rays, showing an increase in the wave-length of a spectrum line when it is scattered.

the character of the radiation as the secondary rays were formed. This conclusion has been experimentally confirmed in the X-ray region by Laub,[2] Gray,[3] Compton,[4] and Crowther,[5] and in the region of γ-rays by Florance[6] and Compton.[7]

Recent spectroscopic measurements by the writer show that the secondary rays have suffered a distinct change in wave-length. Thus in Figure 6 the spectrum of the molybdenum rays after being scattered at 90° by graphite show the K lines at angles distinctly greater than those at which they occur for the primary beam.

Absorption measurements indicate that the secondary rays are composed of two parts, one having the same wave-length as the primary

[1] J. A. Gray, Phil. Mag. 26, 611 (1913).
[2] J. Laub, Ann. d. Phys. 46, 785 (1915).
[3] J. A. Gray, Frank. Inst. Jour., Nov., 1920, p. 643.
[4] A. H. Compton, Phys. Rev. 18, 96 (1921); Nature, 108, 366 (1921).
[5] J. A. Crowther, Phil. Mag. 42, 719 (1921).
[6] D. C. H. Florance, Phil. Mag. 27, 225 (1914).
[7] A. H. Compton, Phil. Mag. 41, 749 (1921).

beam, and the other a slightly greater wave-length.‡ In view of the uncertainty of absorption experiments in deciding such a question, we shall have to wait for more precise spectroscopic measurements before we can say definitely whether any of the secondary rays have the same wave-length as the primary rays. Preliminary spectroscopic work, however, seems to support the absorption measurements in showing the existence of both the unchanged and the longer wave-lengths in the secondary beam. In any case the spectrum shown in Figure 6 leaves no doubt but that a large part of the secondary X-rays have suffered a real change in wave-length. According to the writer's absorption measurements, over the range of primary rays from .7 to .025, Å. U., the wave-length of the secondary X-rays at 90° with the incident beam is roughly 0.03 Å. U. greater than that of the primary ray which excites it.

Interpretation of the softening.—Three possible explanations of this effect have been suggested. Laub* and Crowther° account for the greater effective wave-length of the secondary beam on the assumption that there exist homogeneous fluorescent radiations characteristic of the radiator, but of higher frequency than the characteristic K radiations ("characteristic J-radiation"). Gray§† and Florance¶ conclude that the X-rays are truly scattered, but in the process of scattering are so modified as to become less penetrating. Compton has suggested‡‖ that the primary rays excite in the radiator a type of "general fluorescent radiation," similar in character to the general or "white" radiation emitted by an X-ray tube.

Analogy with the characteristic K and L radiations seems to be the chief reason for Laub and Crowther's view that the observed softening of the secondary ray is due to a fluorescent radiation characteristic of the radiator. The more exhaustive work of Sadler and Mesham+ had indicated, however, that these softer components of the secondary beam increased continuously in hardness with increasing hardness of the primary beam over a wide range. This result has been verified by Gray† and Compton‡‖, who have found that the secondary radiation not only from carbon but also from such elements as aluminium, copper, tin, and lead, may have any wave-length between 0.7 and 0.04 Å. U., according to the wave-length of the exciting rays and the angle at which the secondary rays are studied.[1] Evidence that the secondary radiation is in any way characteristic of the particular element employed as radiator is thus completely lacking.

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* See Note 2, page 16.
° See Note 5, page 16.
§ See Note 1, page 16.
† See Note 3, page 16.
¶ See Note 6, page 16.
‡ See Note 4, page 16.
‖ See Note 7, page 16.
+ See Note 4, page 15.

[1] It is hoped to publish in the near future a full account of the experiments in the X-ray region of which reference 4, page 16 is a brief abstract.

Gray * has shown that if the primary rays consist of thin pulses, as suggested by Stokes' theory of X-rays, and if these rays are scattered by atoms or electrons of dimensions comparable with the thickness of the pulse, the thickness of the scattered pulse will be greater than that of the incident pulse. Thus the effective wave-length of the secondary rays will be greater than that of the primary beam. Since it has been shown that the sharp upper limit to the frequency of the primary X-ray beam from a tube operated at constant voltage is inconsistent with the pulse theory of X-rays,[1] this scattered pulse hypothesis is difficult to defend. But such an explanation has been definitely eliminated by Compton's observation[2] that X-rays which are reflected from a crystal, and which are therefore known to come in long trains of waves, excite in light elements a secondary radiation which is softened to about the same degree as that excited by ordinary X-rays.

If the incident X-rays are homogeneous, as in the writer's experiment, the scattered rays must be homogeneous and of the same wave-length † unless a Doppler effect is present. But in order to account for the observed softening of the secondary rays as due to a Doppler effect, the scattering particles would have to be moving in the direction of the primary beam at a speed comparable with that of light. This is not possible on the classical theory, which supposes that all the electrons in the radiator are effective in scattering. Thus the classical electrical theory appears irreconcilable with the view that the part of the secondary rays that are of greater wave-length than the primary beam are truly scattered. The assumption of a general fluorescent radiation is the obvious and apparently the only alternative. On this view, only that part of the secondary radiation whose wave-length is identical with that of the primary beam is truly scattered, and that of greater wave-length is fluorescent. If this result be correct, it is of far-reaching importance, since as a consequence of their neglect of this type of fluorescence little weight could be attached to the quantitative measurements of scattering made by the earlier investigators (see, however, note 2, page 4).

The Doppler Effect in Secondary X-rays.—On the basis of the quantum theory a different hypothesis may be formed. Let us suppose that each electron when it scatters X-rays receives a whole quantum of energy and reradiates the whole quantum in a definite direction. The momentum which the scattering electron receives from the radiation will then be $h\nu/c$, where h is Planck's constant, ν is the frequency and c is the velocity of light. This will result in a velocity in the forward direction which

[1] Cf. D. L. Webster, Phys. Rev. 7, 609 (1916).
[2] A. H. Compton, Nature, 108, 366 (1921).
* See Note 3, page 16. † See Note 5, page 16.

will produce a Doppler effect as the scattered rays are observed at different directions. In addition, as the electron radiates a quantum of energy toward the observer, the conservation of momentum principle demands that the electron shall recoil with a momentum $h\nu'/c$, where ν' is the average frequency of the scattered radiation. For cases in which the resulting velocity of the electron is small compared with the speed of light, it can be shown on this basis that the ratio of the average frequency of the rays scattered at 90° to that of the incident rays should be $1-h/mc\lambda$. In the case of the molybdenum $K\alpha$ line ($\lambda = .708$ Å) this calculated ratio is 0.966, while the value of the ratio taken from the experiments shown in figure 6 is 0.969. This close numerical agreement would suggest that we should consider scattering as a quantum phenomenon instead of obeying the classical laws of electricity as assumed in the first section of this report.

The view that much of the secondary radiation comes from electrons moving at high speed is supported by the apparent Doppler effect observed in the case of secondary γ-rays. The writer found[1] that when hard γ-rays from RaC were used, the secondary rays at 135° with the primary beam had a wave-length of about 0.08 Å. U., while those at less than 20° were only very slightly softer than the incident rays. Using the value 0.025 Å. U. for the effective primary wave-length,[2] the wave-length of the secondary rays at 20° may be taken as about 0.03 Å. U. According to the Doppler formula,

$$\frac{\lambda_1}{\lambda_2}=\frac{1-\beta\cos\theta_1}{1-\beta\cos\theta_2},$$

the observed wave-lengths at the two angles 20° and 135° indicate a velocity of the source of $\beta=52$ per cent the speed of light, if we suppose with Rutherford[3] that the secondary β-rays are initially ejected in the forward direction. This rapid motion of the radiator will result in a greater radiant energy in the forward than in the backward direction. A comparison of the observed assymmetry of the energy of the secondary γ-rays with that calculated for different velocities of the radiator leads to a value[4] of β = about 55 per cent the speed of light. Thus the asymmetry of both the wave-length and the energy of the secondary γ-rays indicates radiation from a particle moving with slightly over half the speed of light. This is in good accord with Eve's observation[5] that the aver-

[1] A. H. Compton, Phil. Mag. 41, 749 (1921).

[2] A. H. Compton, Phil. Mag. 41, 770 (1921). This value of the wave-length is confirmed by Ellis' recent determination of the wave-lengths of the γ-rays from RaC from the magnetic spectra of secondary β-rays (Proc. Roy. Soc. A., 101, 6, 1922).

[3] E. Rutherford, "Radioactive Substances, etc." p. 276.

[4] A. H. Compton, Phil. Mag. 41, 767 (1921).

[5] Eve, Phil. Mag. 8, 669 (1904).

age speed of the secondary β-rays excited by the hard γ-rays from radium is somewhat greater than half the speed of light.

Polarization.—The view that these softened secondary X-rays are really scattered is apparently confirmed by a study of their polarization. Barkla, in his classic measurement of the polarization of secondary X-rays,[1] found that at 90° the secondary rays from carbon were approximately 80 per cent polarized. The remaining 20 per cent might be accounted for in part by experimental error, and the remainder Barkla ascribed as due possibly to a real lack of polarization because of forces acting on the electrons. Recent experiments by Compton and Hagenow[2] however, have shown that, when the multiple scattering within the radiator is eliminated, the polarization of the secondary X-rays is complete within an experimental error of about 1 per cent. Thus if any fluorescent radiation exists, it must be nearly completely polarized.

Conclusion.—While these properties of the secondary radiation are in accord with the view that it is truly scattered, one must not lose sight of the fact that the classical electron theory demands that scattered rays shall be of the same wave-length as the primary. Moreover, the very satisfactory explanations of excess scattering and of X-ray crystal reflection that have been given require that a considerable number of electrons shall co-operate in their scattering in their proper phases. Such co-operation is contrary to the quantum hypothesis of scattering which has been introduced to account for the change in wave-length of the secondary rays. But if we adhere to the classical theory, we must invoke a general fluorescent radiation to account for the observed softening. Both this suggestion of general fluorescent radiation and the idea of quantum scattering are difficult to reconcile with the very small intensity of the secondary radiation observed at small angles with the incident beam (supra, p. 14), since neither hypothesis supplies the mechanism necessary for destructive interference.

Thus we see that the classical electrodynamics succeeds in explaining quantitatively many of the phenomena of secondary X-radiation, supposing that a considerable part of this radiation is truly scattered. The change in wave-length of the rays as they are transformed from primary to secondary X-rays seems to be in accord rather with quantum principles. But it has not been possible to account on either basis for all the observed phenomena. The theory of secondary X-rays is thus at present in an unsatisfactory form. The close overlapping of the classical and the quantum principles as applied to this problem, however, suggests that here may be a most profitable field for studying the connection between these two points of view.

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[1] C. G. Barkla, Proc. Roy. Soc. 77, 247 (1906).
[2] A. H. Compton and C. F. Hagenow, Phys. Rev. 18, 97 (1921).

II. Photoelectrons Excited by X-Rays

It was observed by Perrin[1] and by Curie and Sagnac,[2] early in the history of X-rays, that when these rays fall on solid screens a type of secondary radiation is emitted which is nearly completely absorbed in 1 mm. of air. Dorn[3] showed that this radiation consisted of negatively charged corpuscles which could be deflected by a magnetic field; and assuming the same ratio of e/m as that of the cathode rays, he found that the velocities of these secondary particles were of the order of 1/10th the velocity of light. We shall call these high speed electrons "photo-electrons," whether liberated by the action of light, X-rays or γ-rays.

Methods of Experimental Investigation

The presence of these photoelectrons can be detected by allowing X-rays to fall on a plate insulated in a good vacuum. The plate is then found to acquire a positive charge, due to the emission of the secondary electrons. The effect is thus strictly analogous to the photoelectric effect.

A second method of investigation is to make use of the ionization produced by the photoelectrons. Thus, it is found that if X-rays strike a solid substance placed in a gas, as in Figure 7, the ionization in the neighborhood of the solid is much more intense than that elsewhere in the gas. The region of intense ionization, being determined by the range of the photoelectrons, may be varied by changing the pressure of the gas. Thus, since the ionization due to the absorption of the X-rays in the gas is proportional to the pressure P, the total ionization I, if the secondary electrons are completely absorbed, is given by

$$I = CP + I_e,$$

where the constant of proportionality C can be determined by experiment, and I_e represents the ionization due to the photoelectrons from the solid. Thus

$$I_e = I - CP.$$

Theoretically this method is open to the objection that it does not distinguish between photoelectrons and secondary X-radiation of very soft type. Under ordinary conditions, however, the ionization due to the electrons is so much greater than that due to the very soft secondary X-rays that no confusion is apt to arise. This method is a convenient one, and has been much used.

In many respects the most satisfactory method of studying these secondary electronic rays is the beautiful one employed by C. T. R. Wilson,[4] in which the tracks of the individual particles are rendered

[1] Perrin, Ann. de Chim. et Phys. (7), vol. 2, p. 496 (1897).
[2] Curie & Sagnac, Jour. de Phys. (4), vol. 1, p. 13 (1902).
[3] Dorn, "Lorentz Jubilee Volume," p. 595 (1900).
[4] C. T. R. Wilson, Proc. Roy. Soc. 87, 277 (1912).

visible by condensing water droplets on the ions formed along their paths. By this means it is possible to count accurately the number of secondary electrons emitted, study their distribution, and make measurements of their range in air. If two simultaneous photographs are taken at right angles with each other, by the method described by Shimizu,[1] the exact shape and total length of the paths may also be determined.

For investigating the velocities of the photoelectrons excited by X-rays, the method of photographing their magnetic spectrum has given the best results. For this purpose, the arrangement employed first by Robinson and Rawlinson[2] is very satisfactory. This arrangement, suggested by Rutherford and Robinson's[3] method of photographing the magnetic spectra of primary β-rays, is illustrated in Figure 8.

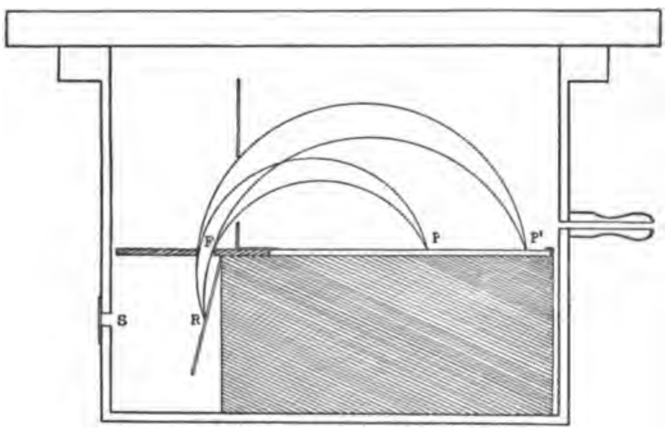

FIGURE 8. Magnetic photoelectron spectrometer. Photoelectrons leaving the radiator R with different speeds are bent by the magnetic field to different points PP^1 on the photographic plate.

A flat, air-tight, brass box having a window S for the admission of the primary X-rays, is evacuated and placed between the poles of a large electromagnet. Secondary electrons from the radiator R go out in all directions, and those passing through the slit F have their paths bent around by the magnetic field to some point P on a photographic plate. The geometrical arrangement is such that all electrons emitted with the same speed from a certain point on R, and passing through the slit F will fall on the same line at P. From the position of this line the radius of curvature can be determined, and the velocity of the electrons responsible for the line may be calculated from the formula,

$$v = RH\frac{e}{m},$$

[1] T. Shimizu, Proc. Roy. Soc. 99, 425 (1921).
[2] Robinson & Rawlinson, Phil. Mag. 28, 277 (1914).
[3] Rutherford & Robinson, Phil. Mag. 26, 717 (1915)

where R is the radius of curvature, H is the effective strength of the magnetic field, and e and m have their usual significance. A very good description of the details of this method is given by de Broglie in his paper on "The Corpuscular Spectra of the Elements."[1]

Distribution of the Photoelectrons

Longitudinal Asymmetry.—Mackenzie[2] first showed that the photoelectrons excited by γ-rays traversing a thin plate are much stronger in the direction of the γ-ray beam than in the opposite direction. The same effect is observed, though to a less degree, in the case of the photoelectrons produced by X-rays[3] and those produced by ultra-violet light.[4] The difference in the amount of the emergence and the incidence photoelectric emission from various metals as excited by γ-rays has been examined by Bragg.[5] He found that for light substances the emergent radiation is very much greater than for the incident (by a factor as great as 20 for the photoelectrons excited in carbon by hard γ-rays). For heavy metals the difference is far less marked, probably because of the much greater scattering of electronic rays by the heavy atoms. Rutherford[6] considers the experiments in accord with the view that the particles initially escape in the direction of the incident γ-rays. It is found that when X-rays are used, the degree of asymmetry does not differ for thin screens of different substances, and that the physical state has no appreciable effect.[8]

As has just been shown, the degree of asymmetry is a function of the wave-length of the incident radiation. However, Bragg finds* the asymmetry to be nearly the same with soft γ-rays as with hard γ-rays. Also Cooksie,[7] Stuhlmann† and Owen[8] find that the degree of asymmetry of the secondary radiation is practically the same, about 7 per cent greater in the forward direction, for wave-lengths varying from 3000 to 0.5 Å. U. There must therefore exist a region of wave-lengths between 0.5 and 0.05 Å. U. where the asymmetry of the photoelectric emission rapidly increases with decreasing wave-length.

The experiments so far considered were performed by ionization methods similar to that described above. It is rather surprising that photographs taken by Wilson's method show no appreciable asymmetry in the direction of the X-ray beam of the photoelectrons

[1] M. de Broglie, Jour. de Phys. et Rad. 2, p. 265 (1921).
[2] Mackenzie, Phil. Mag. 14, p. 176 (1907).
[3] C. D. Cooksie, Nature, 77, 509 (1908).
[4] O. Stuhlmann, Nature, May 12, 1910; Phil. Mag. 22, 854 (1911). R. D. Kleeman, Nature, May 19, 1910; Proc. Roy. Soc. (1910).
[5] W. H. Bragg, Phil. Mag. 15, 663 (1908).
[6] E. Rutherford, "Radioactive Substances and Their Radiations," p. 276 (1913).
[7] C. D. Cooksie, Phil. Mag. 29, 37 (1912).
[8] E. A. Owen, Phys. Soc. Proc. 30, 133 (1918).
* See Note 5. † See Note 4.

liberated in air. The effect does appear when the X-rays traverse a plate of copper (Fig. 7), though here, as Mr. Wilson has pointed out in conversation with the writer, the predominance in the forward direction appears to be due to a bending of the paths of the electrons rather than to an initial asymmetry in their emission.

In order to account for the asymmetrical distribution of the photoelectrons, W. H. Bragg at one time suggested[1] that the primary X-rays exciting these electrons may consist of uncharged material particles, traveling with a speed approaching that of light. He supposed that these "neutrons," on colliding with electrons, transferred to them their energy and momentum. This would obviously account in a qualitative manner for the observed asymmetry of the electronic rays. The discovery † of the same type of asymmetry in the photoelectrons emitted under the action of ultra violet light, however, made this view difficult to defend,[2] and the later discovery of the interference and diffraction of X-rays has made the neutron hypothesis untenable.

An alternative explanation of the phenomenon has been put forward by Richardson.[2] He supposes that as the electron absorbs a quantum $h\nu$ of energy, the momentum of the absorbed radiation is also transferred to the electron, causing a resultant motion in the forward direction. If the kinetic energy after emission is $\frac{1}{2}mv^2 = h\nu$, the average ratio of the forward component of the velocity to the total velocity is shown to be

$$\frac{u}{v} = \frac{1}{2}\frac{v}{c}.$$

When v approaches the velocity of light c, the ratio u/v approaches $\frac{1}{2}$,* which indicates a decided preponderance in the forward direction, whereas for electrons liberated by ultra-violet light the ratio u/v is only about 1/500. This view thus accounts qualitatively for the observed increase in asymmetry of the photoelectric emission with increase in frequency of the exciting radiation. It seems to indicate, however, a greater and more uniform variation of asymmetry with wave-length than is actually observed.

It is interesting to note that the region of wave-lengths 0.5 to 0.05 Å. U. in which the asymmetry of distribution of the photoelectrons rapidly increases is just the region within which the secondary X-radiation becomes strongly asymmetrical. This suggests that both phenomena

[1] W. H. Bragg, Nature, Jan. 23, 1908; Phil. Mag. 16, 918 (1908).

[2] O. W. Richardson, Phil. Mag. 25, 144 (1913); The Electron Theory of Matter, pp. 478–481 (1914).

*This result is obtained without correcting for the variation of mass with velocity. When this correction is made, the limiting value of u/ν for very high frequencies becomes unity. That is, the photoelectrons should be emitted in the direction of the incident rays.

† See Note 4, page 23.

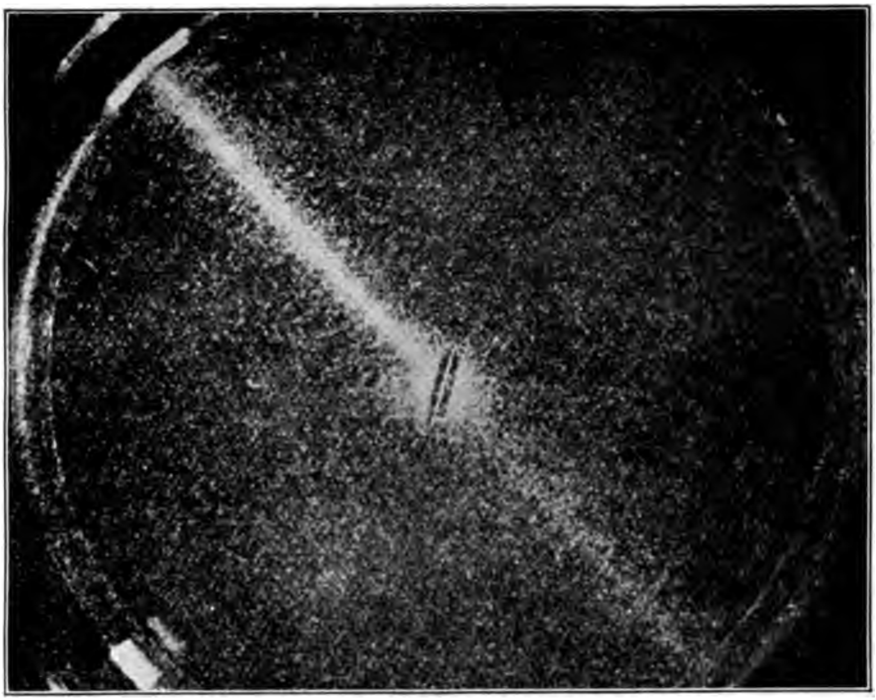

FIGURE 7. C. T. R. Wilson's photograph of X-rays traversing a thin plate of copper, showing the photoelectrons ejected from the copper and the absorption of the X-rays on passing through.

FIGURE 9. Path of X-rays through air, showing tracks of the photoelectrons produced.

may have a common origin, and tends to support the view that the secondary X-rays are emitted by electrons which are moving forward at high speed.

Lateral Asymmetry.—An examination of the electronic ray tracks as photographed by Wilson seems to reveal a marked tendency for these particles to be ejected nearly perpendicular to the direction of propagation of the X-ray beam. For example Figure 8 shows 31 tracks[1] whose point of origin can be distinguished with some certainty. Measurements on this photograph show 3 tracks starting within 45° of the horizontal, 7 whose initial direction is close to 45°, and 21 which start within 45° of the vertical. If the direction of ejection were a matter of chance, the horizontal and vertical directions would from considerations of symmetry be equally probable. Thus it appears that most of the photoelectrons are ejected nearly in the plane of the electric and magnetic vectors of the incident X-rays.

It would be of great interest to examine by this method the initial direction of the electrons ejected by polarized X-rays. It seems probable that the direction of ejection should be close to that of the electric vector.

Velocity of the Photoelectrons.—The maximum velocity of the photoelectrons liberated from a metal is given by the well known photoelectric equation

$$\frac{1}{2}mv^2 = h\nu - w_0, \tag{15}$$

where h is Planck's constant, ν is the frequency of the incident rays, and w_0 is the work done by the electron in escaping from the metal. This equation was first proposed by Einstein[2] as a deduction from the view that radiant energy occurs in discrete quanta, but was shown by Richardson[3] to be a direct consequence of Planck's radiation formula. As applied to light, its accuracy has been established by the researches of Richardson and K. T. Compton,[4] Hughes,[5] and particularly by the accurate experiments of Millikan[6] and his collaborators. In the X-ray region the studies of Innes,[7] Sadler,[8] Beatty,[9] Whiddington[10] and Moseley[11]

[1] This figure is C. T. R. Wilson's photograph, Proc. Roy. Soc. 87, plate 8, No. 4 (1912). I believe the lateral assymmetry of the photoelectrons here referred to has been noticed by Wilson, though I am unable to find any published statement to this effect.

[2] A. Einstein, Ann. d. Phys. 17, 145 (1905).

[3] O. W. Richardson, Phys. Rev. 34, 146 (1912); Phil. Mag. 24, 570 (1912).

[4] O. W. Richardson & K. T. Compton, Phil. Mag. 24, 575 (1912).

[5] A. L. Hughes, Phil. Trans; A, 212, 205 (1912).

[6] R. A. Millikan, Phys. Rev. 7, 18 & 355 (1916).

[7] P. D. Innes, Proc. Roy. Soc. 79, 442 (1907).

[8] Sadler, Phil. Mag. 19, 337 (1910).

[9] Beatty, Phil. Mag. 20, 320 (1910).

[10] Whiddington, Proc. Roy. Soc. 86, 360, 370 (1912).

[11] H. G. J. Moseley, Phil. Mag. 27, 703 (1914).

taken together showed[1] that the *maximum* energy of electrons liberated by X-rays of frequency ν is given very closely by

$$\frac{1}{2}mv^2 = h\nu. \qquad (16)$$

This is evidently in accord with the photoelectric equation, since the fastest electrons will come from the surface of the atom, where w_0 is negligible as compared with $h\nu$ for X-rays.

It is thus apparent that the fastest secondary electrons are emitted with kinetic energy equal to one quantum of the incident radiation. We are, however, equally interested in the energy relations of the slower electrons. Barkla and Shearer[2] were led to the conclusion that all the secondary electrons, whatever their origin, have on leaving the atom the speed corresponding to a whole quantum of the incident radiation. This conclusion was, however, shown by Richardson[3] to be questionable from a theoretical standpoint. Recently Simons,[4] from a study of the absorption of the secondary electrons in thin screens, suggested that different groups of velocities were present, corresponding to different energy losses by the electrons ejected from different parts of the atom. Finally a series of beautiful experiments by de Broglie[5] in the X-ray region and by Ellis[6] in the region of γ-rays, both using the magnetic spectrum method, has shown that at least a large part of the electrons emitted from different energy levels within the atom absorb one quantum of the incident energy, and emerge with their kinetic energy diminished only by the work required to leave the atom. For photoelectrons with these high velocities de Broglie finds it necessary to express the kinetic energy according to the relativity formula

$$T = 1/2m_0v^2\left(1+\frac{3}{4}\beta^2+\frac{5}{8}\beta^4+\cdots\right), \quad \left[\beta=\frac{v}{c}\right] \qquad (17)$$

where m_0 is the electron's mass at low speeds. The photoelectric equation as applied to high frequency radiation thus assumes the form

$$T = h\nu - w_P \qquad (18)$$

where ν is the frequency of the radiation which gives rise to the photoelectron and w_P is the energy required to remove the electron from its initial position in the P (K, L, M or N) energy level.

The experimental evidence indicates that the kinetic energy T, calculated by equation (18), is the maximum which may be possessed

[1] O. W. Richardson, "Electron Theory of Matter," Chapter XIX.
[2] Barkla and Shearer, Phil. Mag. 30, 745 (1915).
[3] O. W. Richardson, Proc. Roy. Soc. 94, 269 (1918).
[4] L. Simons, Phil. Mag. 4, 120, (1921).
[5] M. de Broglie, C. R. 172, pp. 274, 527, 746 & 806 (1921); Journ. de Phys. & Rad. 265 (1921).
[6] C. D. Ellis, Proc. Roy. Soc. 99, 261 (1921).

by an electron ejected from the P energy level. The magnetic spectrum lines of the electronic rays as obtained by de Broglie, though relatively sharp on the high velocity side, shade off gradually on the side of the low velocities. In spite of the fact that thin foils were used, this shading is doubtless due in part to the loss in energy by some of the electrons before they reach the surface of the foil. Whether any electrons possess a smaller amount of kinetic energy as they leave the atom is therefore not answered by these experiments.

The fact should be mentioned that in the experiments of both de Broglie and Ellis a consistent tendency was noticed for the energy losses w_P to appear somewhat greater than the energy corresponding to the P energy level in the atom. This effect was most pronounced when w_P was small. It is possible, however, that this difference is due to consistent experimental errors, and the fact that de Broglie failed to notice such a difference in the experiments which he considered most reliable makes its existence appear questionable. In any case it may be stated that equation (18) holds within 2 or 3 per cent for the swiftest electrons leaving each energy level in the atom.

It is remarkable that in Ellis's paper no magnetic spectrum lines are recorded which are due to the expulsion of electrons from the outer rings. This result is confirmed by de Broglie's photographs, though since in his experiments X-rays are employed, the number of photo-electrons ejected from the outer rings is much smaller than that from the inner rings. In Ellis's experiments, however, γ-rays were employed, so that the greater part of the absorption was due to these outer electrons, as is shown by the fact that both the absorption per atom[1] and the number of electrons ejected per atom[2] is for the γ-rays nearly proportional to the number of electrons in the atom. We should therefore have expected these outer electrons, for which w_P is negligible compared with $h\nu$, to give rise to a strong line for which $T=h\nu$. Ellis finds, however, lines due only to the electrons in the K-rings, and these only for the heavy elements.

If, as has been suggested above, an electron scatters a whole quantum of energy at a time, and receives the momentum of the incident quantum, the average momentum in the forward direction of an electron which has scattered rays of wave-length λ is

$$M=\frac{h\nu}{c}=\frac{m_0 v}{\sqrt{1-\beta^2}},$$

where $\beta=v/c$. Thus the velocity of the electron is given by

$$\beta^2=1/(1+m^2c^2\lambda^2/h^2).$$

[1] Cf. M. Ishino, Phil. Mag. 33, 140, 1917; also infra, p. 45.
[2] Cf. infra, pp. 28 and 29.

For gamma rays with an effective wave-length of 0.025 Å.U., this means a forward velocity of about .7 the speed of light. Obviously such electrons should appear as photoelectrons, though with a velocity much less than that of an electron possessing a whole quantum of kinetic energy. It does not seem unreasonable to suppose that it may be such scattering electrons which constitute the photoelectrons excited by γ-rays in the lighter elements. The fact that the number of such electrons should be proportional to the atomic number and that the calculated velocity is of the observed order of magnitude confirms this view. Thus Ellis's failure to observe photoelectrons with the maximum kinetic energy $h\nu$ seems to support the hypothesis that most of the secondary β-rays excited by γ-rays are not a result of fluorescent absorption, but are rather a by-product of the scattering process.

Range of the Photoelectrons.—This question is identical with that of the range of the primary cathode rays, which has been discussed in an earlier portion of this report.[1] In one respect the problem is simplified however: it is possible to photograph the track of the electron by Shimizu's method, and to learn definitely both the distance to which the particle can penetrate, and the total length of its irregular path. While the former quantity is found to differ greatly for different rays, there is no evidence that the total length of path differs appreciably for electrons starting with the same initial velocity.

Number of Photoelectrons.—A series of experiments by W. H. Bragg,[2] Barkla[3] and their collaborators suggested strongly that the true absorption (as opposed to scattering) of X-rays is due solely to the excitation of the secondary electronic rays. This conclusion was supported by C. T. R. Wilson's photographs[4] of the path of an X-ray beam through air, which showed no ionization along the path of the X-rays except that due to the action of the high speed electrons which were liberated. On this view, X-ray energy can be dissipated in only two ways, either by scattering or by the excitation of photoelectrons.

It is possible that there may be two types of photoelectrons, those whose liberation excites the characteristic fluorescent radiation, and those which recoil after scattering a quantum of energy. According to the results of de Broglie and Ellis, each electron of the first type represents the absorption of one quantum of energy $h\nu$ from the primary beam. If the second type of photoelectron exists, it also should represent a whole quantum of energy, the greater part of which appears as scattered radiation and the remainder as kinetic energy of the recoiling electron. The energy of the second type will ordinarily be small compared with that of

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[1] Bulletin of the National Research Council, vol. 1, p. 424 (1920).
[2] W. H. Bragg, Phil. Mag. 20, 385 (1910) et al.
[3] C. G. Barkla, Phil. Mag. 20, 370 (1910) et al.
[4] C. T. R. Wilson, Proc. Roy. Soc. 87, 288 (1912).

the first type. The evidence is thus consistent with the view that each photoelectron represents the removal of one quantum of energy from the primary beam, and that no other energy is lost except perhaps through true scattering.

It follows as a result of this conclusion that the energy absorbed per centimeter path of the X-ray beam should be

$$Nh\nu = E_i\tau,$$

where N is the number of β-particles liberated, E_i is the energy of the X-rays incident upon the substance and τ represents the fluorescent absorption coefficient (cf. infra, p. 37). Thus the number of photoelectrons liberated per centimeter path of the X-ray beam should be

$$N = E_i\tau/h\nu. \tag{20}$$

Partial support of this relation (20) is given by the fact that the number of electrons ejected from an atom is independent of its state of chemical combination,[1] as is also the energy absorbed by the atom. Moore has shown[1] also that the number of photoelectrons emitted by different light atoms traversed by X-rays is proportional to the fourth power of the atomic weight. This corresponds exactly with Owen's law (cf. infra, p. 38) that the fluorescent absorption per atom under similar circumstances is proportional to the fourth power of the atomic number. It follows therefore that the number of photoelectrons is proportional to the X-ray energy truly absorbed, as stated by equation (20). Although no direct experimental determination of the factor of proportionality has been made, there seems no reason to doubt that this factor is the energy quantum $h\nu$.

In the region of γ-rays the number of secondary photoelectrons emitted per atom is more nearly proportional to the first power than to the fourth power of the atomic number, except for the very heavy elements in which appreciable characteristic X-rays are excited.[2] The number is, however, closely in accord with the total absorption of the γ-rays. Thus Hackett and Eve[2] find that γ-rays traversing plates of lead, iron and aluminium liberate β-particles in the ratio of 1.00 to 0.70 to 0.75 respectively. But Ishino[3] finds the total absorption per electron in these elements to occur in the ratio of 1.00 to 0.76 to 0.77 respectively. This close correspondence can leave little doubt that equation (20) applies to the electrons liberated by gamma-rays as well as those liberated by X-rays. Since these photoelectrons represent energy absorbed from the γ-ray beam, however, it follows that even in

[1] H. Moore, Proc. Roy. Soc. 91, 337 (1915).
[2] Cf. E. Rutherford, "Radioactive Substances, etc." pp. 275–276.
[3] M. Ishino, Phil. Mag. 33, 140 (1917).

the light elements by no means all of the absorbed energy can reappear in the secondary γ-rays, as has occasionally been argued.[1] The result is rather in accord either with the writer's conclusion[2] that the greater part of the energy removed from a γ-ray beam is fluorescently absorbed, or with the quantum conception of scattering suggested previously (p. 18).

Form of the Photoelectron Tracks.—From a study of the electron tracks photographed with his expansion apparatus, C. T. R. Wilson found, "The rays show two distinct kinds of deflection as a result of their encounters with the atoms of the gas—Rutherford's 'single' and 'compound' scattering. The gradual or cumulative deviation due to successive deflections of a very small amount is evidently, however, in this case much the more important factor in causing scattering, all the rays showing a large amount of curvature, while quite a small proportion show abrupt bends. When abrupt deflections occur they are frequently through large angles, 90° or more."[3]

Several people have noticed that the electron tracks in Wilson's photographs show a uniform curvature over considerable distances such as is not to be expected if the deflections are fortuitous. Many of the tracks have the form of converging helices,[4] such as might be due to motion in a strong magnetic field. But the axes of these helices have different orientations for the different electrons (nearly random orientations). It follows that the axis of each helix must be determined by some polarity of the photoelectron whose orientation remains nearly fixed as the particle moves through several centimeters of air and traverses several thousand atoms. This suggested to Shimizu[2] a gyroscopic action of the electron, and the writer has shown[5] that a spinning electron will induce magnetization in the surrounding atmosphere which will result in a strong induced magnetic field at the electron. Using for the angular momentum of the electron the value $h/2\pi$, it is found that one can thus account reasonably satisfactorily for the observed forms of the electron tracks.

An alternative suggestion of the origin of these helical tracks has recently been made by Glasson.[6] He supposes that a chance orientatation of the magnetic molecules in air gives rise to the magnetic fields which result in the curved tracks of the photoelectrons. It follows, however, from Weiss's theory of ferromagnetism that spontaneous magnetisation is not to be expected in a gas; and even should perfect alignment of the molecular magnets occur, the resulting field would be only

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[1] Cf. e.g. Barkla and White, Phil. Mag. 34, 278 (1917).
[2] A. H. Compton, Phil. Mag. 41, 757 (1921).
[3] C. T. R. Wilson, Proc. Roy. Soc. 87, 289 (1912).
[4] Cf. A. H. Compton, Phil. Mag. 41, 279 (1921).
[5] A. H. Compton, ibid., and Frank. Inst. Journ. p. 154 Aug. (1921).
[6] J. L. Glasson, Nature, 108, 421 (Nov. 24, 1921).

a small fraction of that (greater than 1000 Gauss) required to account for the observed curvatures. This suggestion therefore does not help us.

If the writer's explanation of the spiral tracks is the correct one, it is of great interest. For it means that the electron acts as a tiny magnet as well as an electric charge, and that it is a dynamical system, which, by nutational or elastic oscillations, may radiate energy even though separate from an atom.

III. The Absorption of X-rays

If radiation in traversing a thin layer of substance is reduced in intensity by a constant fraction μ per centimeter of the substance traversed, the intensity of the radiation after penetrating to a depth x is

$$I = I_0 e^{-\mu x},$$

where I_0 is the intensity at the surface. The quantity μ is called the "absorption coefficient." Similarly μ/ρ, the "mass absorption coefficient," is the fraction of a beam 1 cm.^{2} cross section absorbed per gram of substance traversed; and μ/ν, the "atomic absorption coefficient," where ν is the number of atoms per cm.,^{3} is the fraction of such a beam absorbed by each atom of the substance.

In order to obtain consistent results in the absorption measurements, the beam of X-rays passing through the absorbing material must be narrow, and the opening into the ionization chamber small, so that no appreciable amount of secondary rays will pass with the primary rays into the ionization chamber. This condition has not always been met in the earlier measurements on the absorption of X-rays and γ-rays, which has made much of this work of doubtful value. Measurements made on the X-rays reflected from crystals, however, have nearly always met this condition.

The following tables give the absorption coefficients of various wave-lengths in different representative elements:

The values here given for lithium, carbon, oxygen, aluminium (H), iron and water (H) are due to Hewlett,[1] and those for aluminium (R), copper, molybdenum, silver, lead and water (R) are due to Richtmyer,[2] except for two measurements on lead due to Hull and Rice,[3] one each on aluminium and copper due to Duane[4] and those for wave-length .025*

[1] C. W. Hewlett, Phys. Rev. 17, 284 (1921).
[2] F. K. Richtmyer, Phys. Rev. 18, 13 (1921).
[3] A. W. Hull and M. Rice, Phys. Rev. 8, 836 (1916).
[4] W. Duane, Proc. Nat. Acad., March, 1922.
* $\lambda = .025$ represents the γ-rays from RaC, using the wave-length as measured by A. H. Compton, Phil. Mag. 41, 770 (1921). This value of the effective wave-length is confirmed by Ellis's recent results (Proc. Roy. Soc. A 101, p. 6, 1922). He finds homogeneous gamma-rays from RaC of wave-lengths .045, .025, .021 and .020 A. Line .020 is the strongest.

TABLE I. *Mass Absorption Coefficients*, μ/ρ.

Å.U.	Li 3	C 6	N 7	O 8	Al 13 (H)	Al 13 (R)	Fe 26	Cu 29	Mo 42	Ag 47	Pb 82	H_2O (H)	H_2O (R)
.025	....	....	....	....	.071		.068	....	....	....	.076	....	....
.100	....	.144	....	....	.162	<—Duane	—>	.32	....	....	....	....	.168
.125	....	.154	....	.146	.174	.181	.399	.46	1.35	.... Hull—>	3.0	.161	.176
.150	....	.163	.163	.163	.200	.198	.585	.79	1.96	....	1.57	.180	.185
.175	....	.166	.171	.174	.236	.221	.820	1.13	2.83	3.69	2.55	.195	.195
.20	....	.173	.177	.183	.273	.259	1.06	1.56	4.02	6.00	4.60	.199	.204
.25	....	.187	.193	.207	.358	.370	1.88	2.77	7.42	11.4	8.48	.219	.229
.30	.172	.202	.224	.243	.517	.532	3.09	4.50	12.7	18.2	14.2	.246	.261
.35	.188	.219	.251	.289	.719	.764	4.77	6.95	19.1	27.2	22.6	.283	.301
.40	.208	.240	....	.336	.982	1.06	7.02	10.1	26.7	38.6	33.6	.334	.354
.50	.245	.304	....	.488	1.86	1.92	13.86	18.8	48.6	11.5	60.6	.491	....
.60	.306	.394	....	.730	3.05	3.23	22.6	31.6	80.7	19.6	....	.711	....
.70	.403	.532	....	1.08	4.84	5.05	35.3	48.8	18.8	....	....	1.023	....
.80	....	.706	....	1.53	7.26	....	50.7	....	27.2	....	....	1.475	....
1.00	....	1.27	....	....	13.80	....	90.2	....	53.	87.	<—Bragg	2.70	....

TABLE II. *Atomic Absorption Coefficients* $\times 10^{23}$. $\frac{\mu}{\nu} = \frac{\mu}{\rho} \times \frac{W}{N}$.

Å. U.	H 1	Li 3	C 6	N 7	O 8	Al 13	Fe 26	Cu 29	Mo 42	Ag 47	Pb 82	H_2O (H)
.025	....	....	....	....	....	.317	.625	....	....	....	2.60	....
.100	....	....	.285	....	....	.724	....	3.3	....	....	....	....
.125	.04[6]	....	.305	....	.385	.792	3.67	4.8	21.3	....	103.	.478
.150	.05	....	.323	.376	.430	.889	5.38	8.3	31.0	....	53.6	.534
.175	.06	....	.329	.395	.459	1.04	7.55	11.8	44.7	66.5	86.1	.578
.20	.05	....	.343	.409	.482	1.19	9.75	16.4	63.5	107	157	.591
.25	.05	....	.370	.446	.546	1.62	17.3	29.0	117	203	290	.650
.30	.04[5]	.197	.400	.518	.641	2.34	28.4	47.2	201	323	485	.730
.35	.04	.215	.433	.580	.763	3.31	43.9	72.9	302	483	772	.840
.40	.05	.238	.475	....	.886	4.56	64.5	106	422	686	1150	.992
.50	.08	.280	.602	....	1.29	8.44	127	197	769	204	2070	1.458
.60	.09	.350	.780	....	1.92	14.0	208	332	1277	348	....	2.11
.70	.10	.462	1.052	....	2.85	22.1	325	512	297	....	....	3.04
.80	.17	....	1.40	....	4.03	32.4H	466	....	430	....	....	4.38
1.00	....	....	2.51	....	....	61.6H	830	....	838	2000	....	8.01

which are due to Ishino.[1] The writer has interpolated between the values given in the original papers to obtain the values for the wave-lengths desired. For additional data regarding the absorption coefficients of homogeneous X-rays reflected from crystals, the following authorities may be consulted: Bragg and Peirce (Phil. Mag. 28, 626, 1914) give data for the elements Al, Fe, Ni, Cu, Zu, Pd, Ag, Sn, Pt, Au, for wave-lengths between .491 and 1.32 Å.U. Hull and Rice* have studied the absorption by Al, Cu and Pb of short wave-lengths. Glocker (Phys. Zeitshr. 19, 66, 1918) gives a valuable discussion of absorption coefficients on the basis of data at that time available. Owen (Proc.

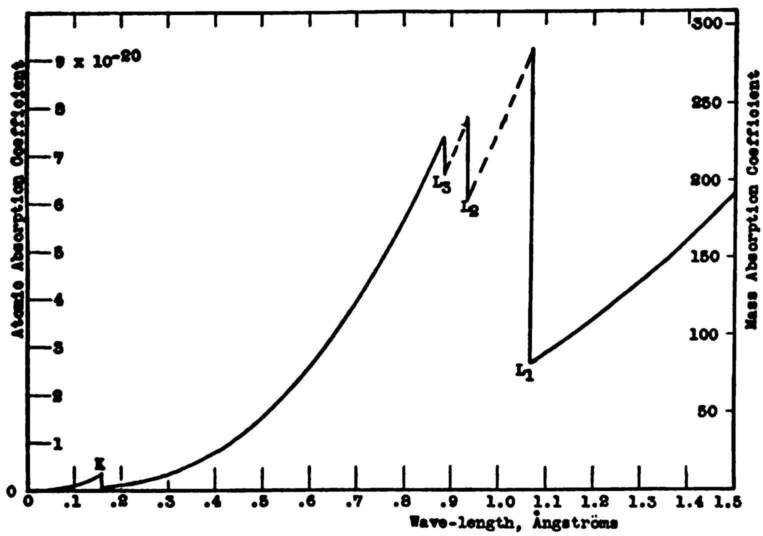

FIGURE 10. Absorption of X-rays by platinum, showing the rapid increase in absorption with increasing wave-length, and the critical *K* and *L* absorption limits.

Roy. Soc. 94, 510, 1918) gives absorption coefficients for 24 different elements of atomic numbers less than 35 for $\lambda = .586$ Å.U. Most of the work by Williams (Proc. Roy. Soc. A, 94, 571, 1918) has been repeated with greater care in the measurements from which Table I has been taken.

The most prominent characteristics of the absorption coefficients as functions of the wave-length and the atomic number are shown in Figures 10 and 11. In Figure 10 is shown the manner in which a given element, in this case platinum, absorbs radiation of different wave-lengths. In general the absorption coefficient increases rapidly with an increase of wave-length. There exist, however, certain critical regions in which for a slightly increased wave-length there is a sudden decrease in absorption. The wave-lengths at which such sudden changes

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[1] M. Ishino, Phil. Mag. 33, 140 (1917).
* See Note 3, page 31.

occur are known as the critical absorption wave-lengths. It is found that if the wave-length of the radiation is shorter than the shortest of these critical wave-lengths, the complete X-ray spectrum of the absorbing element is excited, including the characteristic K-radiation. A slightly longer wave will excite only the characteristic fluorescent L, M, etc. radiations, but not that of the K-type. Similarly there are at least three critical absorption wave-lengths associated with the L series, at each of which a separate portion of the emission spectrum of the L series disappears, until at wave-lengths longer than 1.07 Å.U. no fluorescent L-radiation is excited. Experiment shows that the critical absorp-

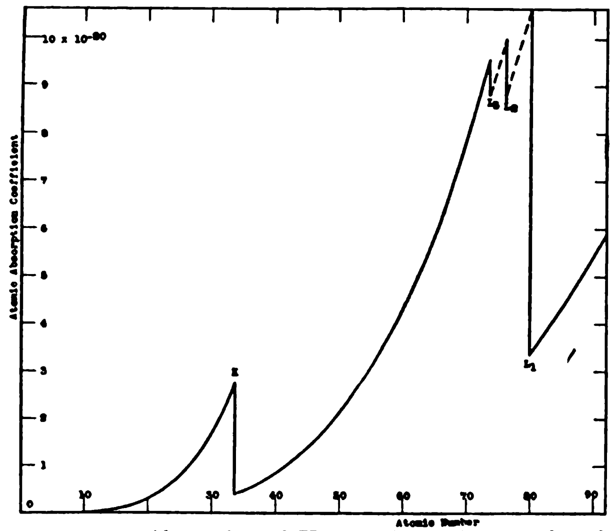

FIGURE 11. Absorption of X-rays of 1 A.U. wave-length per atom of different elements.

tion wave-length associated with any X-ray spectral series is very slightly shorter than the shortest emission wave-length of the series. Thus any element is especially transparent to its own characteristic radiation.

Figure 11 shows the absorption per atom of X-rays of wave-length 1.00 Å.U. in the different elements. The rapid increase of the absorption with the atomic number is prominent. But here again there exist the critical points at which sudden decreases in the absorption occur. Thus arsenic, of atomic number 33, absorbs this wave-length much more strongly than selenium, of number 34, corresponding with the fact that rays of 1 Å.U. wave-length will excite the characteristic K-radiation of arsenic but not of selenium. Similarly there exists a critical atomic number for the L series in the region of platinum (N = 78).

Critical absorption wave-lengths have been observed corresponding not only to the K and L series of the absorber but to its M series as well in the case of the very heavy elements.

Some discussion has arisen with regard to the existence of a critical absorption of a shorter wave-length than the K-radiation, which could be ascribed to a possible "J" radiation. Several experimenters, including Barkla and White,[1] Williams,[2] Owen[3] and Dauvillier,[4] have obtained evidence which they have taken to indicate the existence of such critical wave-lengths; and Laub[5] and Crowther[6] have observed penetrating fluorescent radiation which they have attributed to this source. We have seen above, however (supra, p. 17), that this fluorescent radiation has no definite wave-length characteristic of the radiating element, and is probably ascribable either to a general fluorescent radiation similar to the "white" radiation from an X-ray tube or to scattering by moving electrons. Furthermore, there is no agreement between the values given by different observers for their critical J wave-lengths. The careful measurements of Richtmyer and Grant[7] and those of Hewlett[8] on the absorption of X-rays by light elements have shown no indication whatever of these supposed critical wave-lengths. And finally an examination of the radiation from an X-ray tube with an aluminium target led Duane and Shimizu to conclude[9] that "aluminium has no characteristic lines in its emission spectrum between the wave-lengths $\lambda = .1820$ Å. and 1.259 Å. that amount to as much as 2 per cent of the general radiation in the neighborhood." The evidence is thus strongly against the existence of a characteristic J radiation.

A very satisfactory discussion of these critical absorption wave-lengths, with tables of their values for the different elements, is given by Duane in the first part of this report.[10]

An empirical formula which has been found to express fairly satisfactorily the absorption by all elements of atomic number greater than 5 for wave-lengths between 0.1 and 1.4 Å.U.* is

$$\frac{\mu}{\nu} = KN^4\lambda^3 + .8N\sigma_0 \qquad (21)$$

[1] Barkla and White, Phil. Mag. 34, 270 (1917).
[2] Williams, Proc. Roy. Soc. 94, 567 (1918).
[3] E. A. Owen, Proc. Roy. Soc., 94, 339 (1918).
[4] Dauvillier, Ann. de Phys., 14, 49 (1920).
[5] J. Laub, Ann. d. Phys. 46, 785 (1915).
[6] J. A. Crowther, Phil. Mag. 42, 719 (1921).
[7] Richtmyer and Grant, Phys. Rev. 15, 547 (1920).
[8] Hewlett, Phys. Rev. 17, 284 (1921).
[9] Duane and Shimizu, Phys. Rev. 13, 288 (1919); 14, 389 (1919).
[10] W. Duane, Bull. Nat. Rsch. Coun. 1, 386 (1920).

* This is equivalent to a similar formula employed by Richtmyer,* except that through a typographical error he gives the value of K as 2.29×10^{-27}. The first use of the factors N^4 and λ^3 that I find are by Bragg and Peirce (Phil. Mag. 28, 626, 1914) and by Duane and Hunt (Phys. Rev. 6, 166, 1915) respectively. The term $N\sigma_0$, representing the scattering, was employed by Barkla and Collier (Phil. Mag. 29, 995, 1912); but Hull and Rice,† Hewlett§ and Richtmyer* have found a term equivalent to $.8\ N\sigma_0$ to be more satisfactory, especially at very short wave-lengths with the lighter elements. Glocker (Phys. Zeitsch. 19, 66, 1918) suggests that somewhat better agreement may be obtained if slightly different values of the exponents of N and λ are used, though this is doubtful (cf. Richtmyer*).

* See Note 7, page 31. † See Note 3, page 31. § See Note 1, page 31.

Here λ is the wave-length of the X-rays employed, N is the atomic number of the absorber, K is a universal constant having the value 2.29×10^{-2} for wave-lengths shorter than the critical K absorption wave-length, if λ is expressed in cm., and a value $.33\times 10^{-2}$ when λ is between the critical K and L absorption wave-lengths. The quantity σ_0 is given by the expression

$$\sigma_0=\frac{8\pi}{3}\frac{e^4}{m^2c^4}, \tag{22}$$

and has the value 6.63×10^{-25}. It represents the total energy scattered by a single electron, calculated by integrating Thomson's formula (1) over the surface of a sphere.

The extent of the agreement of this expression (21) with the experimental values for the representative elements carbon, aluminium, iron silver and lead is exhibited for wave-lengths between 0.1 and 1.0 Å. U. in Figure 12. The logarithms of the atomic absorption coefficients are plotted against the logarithms of the wave-lengths. It is remarkable that a formula with but 4 arbitrary constants is able to express so accurately the absorption by some 80 elements of radiation over so wide a range of wave-lengths. It would suggest that the relation is of some physical significance. Nevertheless, the formula is unsatisfactory for extrapolation to shorter wave-lengths, since the minimum absorption that it can give, $0.8\ N\sigma_0$, corresponds to a mass absorption coefficient of about 0.16. This is not in agreement with the mass absorption coefficient about 0.07 observed for all elements when hard gamma rays are employed. A theoretical formula which describes the absorption more satisfactorily is given below (equation 39).

Theory of X-ray Absorption.—The absorption of high frequency radiation is due to at least two independent processes. The more important of these is usually the energy spent in exciting photoelectrons, and resulting in fluorescent radiation. There is always, however, a certain amount of energy removed from the primary beam by scattering. Thus the total absorption coefficient may be written as

$$\mu=\tau+\sigma,$$

where τ represents the "true" or "fluorescent" absorption, and σ the energy lost by scattering.[1]

The Fluorescent Absorption.—This part of the absorption represents the energy which, as we have seen (supra p. 29), is transformed into kinetic energy of the photoelectrons in accord with the quantum rela-

[1] On the classical theory, the energy of the scattered X-rays should equal the energy σ removed from the primary beam to produce the scattered rays. On the quantum theory (supra, p. 18) a part of the energy σ goes into the kinetic energy of recoil of the scattering electrons. In either case the energy σ removed in the process of scattering presumably follows a different law of variation with λ than does the energy τ spent in exciting the fluorescent radiation.

tion, $\frac{1}{2}mv^2 = hc/\lambda$. Experiment shows also,[1] in accord with our empirical formula (21), that it is the sum of a series of terms which may be written very approximately thus:

$$\frac{\tau}{\nu} = [K_K N^4 \lambda^3]_{\lambda<\lambda_K} + [K_L N^4 \lambda^3]_{\lambda<\lambda_L} + \cdots. \tag{23}$$

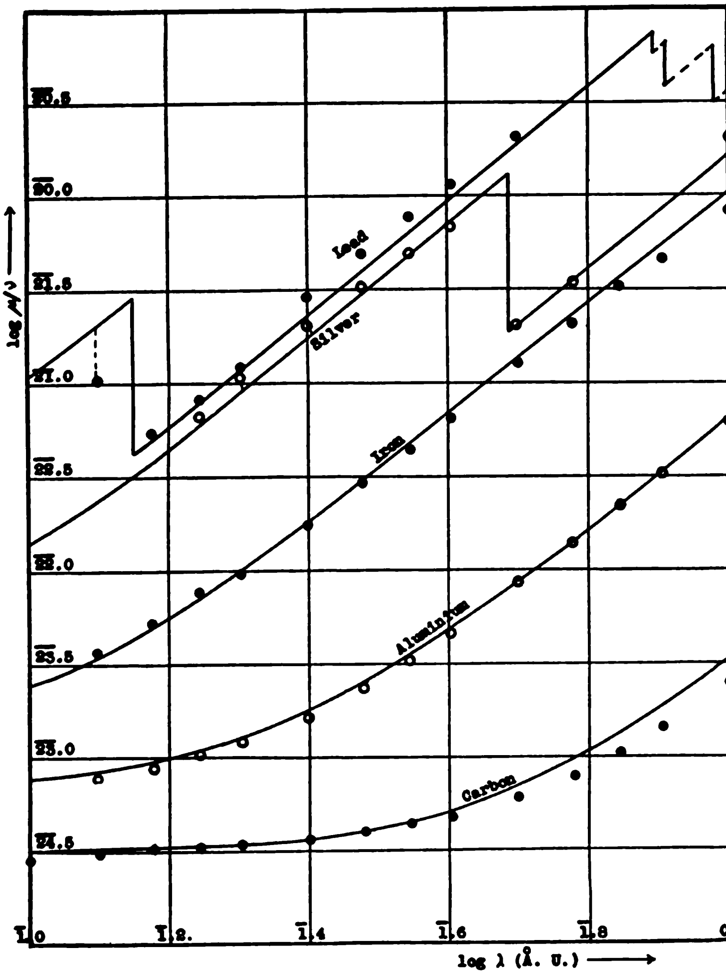

FIGURE 12. Comparison of the experimental values of X-ray absorption for different wave-lengths and atomic numbers with the values predicted by the empirical formula (21).

The first term in this expression presumably represents the absorption by the electrons in the K ring, and is to be used when the incident wave-length λ is less than the critical K absorption wave-length λ_K. Similarly

[1] This result was first reached empirically by E. A. Owen, Proc. Roy. Soc. 94, 522 (1918), and represents a more accurate statement of the absorption law first proposed by him in 1912. (Proc. Roy. Soc. A. 86, 434).

the other terms represent the absorption by the electrons in the *L*, *M*, etc. rings respectively. These facts, especially the reappearance of the absorbed energy in quanta of kinetic energy, would suggest that X-ray absorption is essentially a quantum phenomenon, to which presumably the classical electrodynamics may not be successfully applied. It is therefore surprising to find that quantum principles have offered no suggestion as to the significance of Owen's simple law (23)* whereas the classical electrical theory may be applied to the problem with striking results.

Equation (23) has been derived theoretically by the writer[1] in a simple though uncritical manner, making use of J. J. Thomson's old hypothesis of X-ray pulses. Such a solution of the problem is unsatisfactory, since the basic hypothesis of incident X-rays consisting of very short pulses is inconsistent with the fact that X-ray spectrum lines are very sharp.[2] But the fact that Owen's empirical formula can thus be derived theoretically, strongly suggests that the law may be of real physical significance.

Formal derivation of Owen's law.—Considering the general case in which both the emitting and the absorbing electrons are electrically charged particles capable of executing damped harmonic oscillations, the equation of motion of the absorbing electron when traversed by the incident wave is[3]

$$x = A_1 e^{-r_1 t} \cos (q_1 t + \delta_1) + A_2 e^{-r_2 t} \cos (q_2 t + \delta_2). \tag{24}$$

This motion consists of a forced oscillation whose frequency $q_1/2\pi$ and damping r_1 are those of the incident wave, combined with a free oscillation of the absorbing electron's natural frequency $q_2/2\pi$ and damping r_2. The energy removed from the primary beam in producing this motion is the initial energy of the electron's resulting free oscillation plus the energy spent in executing the forced oscillation against the resistance due to damping.

The energy corresponding to the second term of this expression is derived from the first wave of the incident train. It has therefore the effect of quickly smoothing off the front of the train of waves, so that after passing through a thin sheet of matter the incident wave train no longer starts suddenly, but gradually reaches a maximum and then dies down. A similar smoothing of the wave-front also results from reflection by a crystal. For this reason, in absorption measurements as usually made, the amplitude A_2 is so small as to make the corresponding part of the absorption negligible.[4] The absorption actually mea-

[1] A. H. Compton, Phys. Rev. 14, 249 (1919).

[2] Cf. e.g. D. L. Webster, Phys. Rev. 7, 609 (1918), and G. E. M. Jauncey, Phys. Rev. October (1921).

[3] A. H. Compton, Washington University Studies, 8, 117 (1921).

[4] A more complete discussion of this part of the absorption is given by A. H. Compton, Washington University Studies, 8, pp. 116–120 (1921).

* Cf., however, p. 43.

sured is accordingly due to the motion corresponding to the first term of expression (24) against the resistance due to damping.

The value of the absorption per atom due to the K electrons calculated on this basis may be shown to be,[1]

$$\left(\frac{\tau}{\nu}\right)_K = 8\pi N_K \frac{e^2}{mc} \cdot \frac{r^2}{p_1^2} \Big/ \left\{ \left(1-\frac{p_2^2}{p_1^2}\right)^2 + \frac{4r_1}{p_1^2}(r_2-r_1)\left(1-\frac{p_2^2}{p_1^2}\right) + \frac{4}{p_1^2}(r_2-r_1)^2 \right\}, \qquad (25)$$

where $(\tau/\nu)_K$ is the part of the atomic absorption coefficient due to the N_K electrons in the K ring, e, m and c have their usual values, and $p_1^2 = q_1^2 + r_1^2$; $p_2^2 = q_2^2 + r_2^2$, where the q's and r's are as in equation (24).

Experiment shows* that the damping coefficient r_1 of the primary rays is very small. If it is considered negligible, there are two special cases of interest. First we may assume that the natural period of the absorbing electrons is very great compared with the period of the incident X-rays. Substituting $p_1 = 2\pi c/\lambda$, equation (25) then reduces to

$$\left(\frac{\tau}{\nu}\right)_K = \frac{2N_K}{\pi} \frac{e^2}{mc^2} \frac{r_2\lambda^2}{1+\frac{4r_2^2}{p_1^2}}. \qquad (26)$$

If r_2 is small compared with p_1, this means that the absorption due to the K electrons is equal to a constant times $r_2\lambda^2$. In order for Owen's law to hold, i. e. $\tau/\nu = KN^4\lambda^3$, r_2 must therefore be proportional to $N^4\lambda$. It is accordingly possible to account for the observed absorption of X-rays on this view if the appropriate value of the damping of the absorbing electrons' motion is postulated.

As our second case we may consider the K absorption band to be due to the presence of electrons whose natural frequency may be anywhere between p_K and ∞, where p_K is the angular frequency corresponding to the critical absorption limit. If the damping of these electrons' forced oscillations is small compared with p, and is the same for different wave-lengths, it can be shown that the number of electrons responsible for the K absorption band is related to the absorption coefficient thus:[2]

$$N_K = \frac{mc^2}{\pi e^2} \int_0^{\lambda_K} \left(\frac{\tau}{\nu}\right)_K \frac{d\lambda}{\lambda^2} . \qquad (27)$$

Let us suppose that the absorption is proportional to some power of the wave-length, say to λ^{s+1} Then we may write

[1] A somewhat similar calculation has been performed by R. A. Houstoun, Proc. Roy. Soc. Edinburgh, 40, 34 (1920).

[2] This value for N_K is obtained by a calculation similar to that performed by R. A. Houstoun, Proc. Roy. Soc. Edinburgh, 40, 34 (1920). It is hoped to treat this subject more fully in an early paper.

* See Note 2, page 39.

$$\left(\frac{\tau}{\nu}\right)_K = C_K \lambda^{x+1}$$

for wave-lengths between O and λ_K. Using this value in equation (27) we obtain,

$$N_K = \frac{mc^2}{\pi e^2 x} C_K \lambda_K^x. \tag{28}$$

The unknown quantities x and N_K can be determined as follows. First, choose two elements such that the critical K absorption wave-length λ_{K1} of one is equal to the critical L absorption wave-length λ_{L2} of the other. It follows then from equation (28) that

$$\frac{N_L}{N_K} = \frac{C_{L2}}{C_{K1}}. \tag{29}$$

That is, the ratio of the number of electrons responsible for the L and the K absorption bands is equal to the ratio of the atomic absorption coefficients for the same wave-length by the two elements chosen as specified above.[1] The weighted mean of the three critical L wave-lengths for lead is about 0.90 Å.U. The element with a critical K wave-length of this value should have an atomic number of 35.35. Experiment shows that the L absorption by lead is 4.2±.3 times as great as the K absorption by such an element[2], which means that there are about 4.2 times as many electrons in the L as in the K ring. This result is in exact accord with modern theories of atomic structure, which would assign 2 electrons to the K ring and 8 to the L orbits.

The value of the exponent x may be calculated if two elements are so chosen that the L absorption per atom due to one is equal to the K absorption by the other, i.e. $C_{L2} = C_{K1}$ Then by equation (28),

$$\frac{N_L}{N_K} = \frac{\lambda_{L2}^x}{\lambda_{K1}^x},$$

or

$$x = \log\left(\frac{N_L}{N_K}\right) \Big/ \log\left(\frac{\lambda_{L2}}{\lambda_{K1}}\right). \tag{30}$$

The elements gold and silver satisfy within experimental error the condition imposed. λ_K for silver is .485, and the weighted mean λ_L for

[1] This formula (29) may be derived from much more general assumptions than those used above. It is necessary only to assume that the average absorption by a K electron is equal to that by an L electron if both have the same critical absorption limit. The formula may obviously be extended to the M etc. rings, and gives perhaps the most reliable experimental means now available for estimating the relative number of electrons in these rings.

[2] For the L absorption by lead, use is made of Richtmyer's experimental values, noting according to Bragg and Peirce (X-rays and Crystal Structure, p. 181), that 3/4 of this absorption is due to the L electrons. The K absorption by element 35.35 is calculated from the empirical law (21), remembering that a fraction (2.29−.33)/2.29 of this absorption is due to the K electrons.

gold is about 0.99. Using the experimental value $N_L/N_K = 4.2$, relation (30) becomes,

$$x = \log 4.2/\log\ (.99/.485) = 2.0.$$

Thus within experimental error the absorption should be proportional to the cube of the wave-length.

Equation (28) may now be written in the form

$$C_K = \frac{2\pi N_K e^2}{mc^2 \lambda_K^3} = DN_K \Big/ \lambda_K^3, \tag{31}$$

where

$$D = 2\pi e^2/mc^2. \tag{32}$$

Thus, introducing the similar terms representing the L, M, etc. absorption, we obtain the theoretical absorption law

$$\frac{\tau}{\nu} = D\lambda^3 \left\{ \left[N_K \Big/ \lambda_K^3 \right]^{\lambda<\lambda_K} + \left[N_L \Big/ \lambda_L^3 \right]^{\lambda<\lambda_L} + \cdots \right\}. \tag{33}$$

In view of Moseley's approximate relation, $N^2 \propto 1/\lambda_K$, this is approximately equivalent to Owen's formula. But now the formula does not involve a single arbitrary constant.[1]

The agreement of the experiments with this formula while good is not perfect. The accuracy of the λ^3 and the N^4 relations are exhibited in Figure 11. When $1/\lambda_K^3$ is substituted for N^4, the agreement is of about the same order of accuracy, though small consistent variations appear in the opposite sense as the atomic number of the absorber is varied. According to equation (32), the constant D should have the value 1.77×10^{-12}. Its average experimental value in the K absorption region, taking N_K to be 2, is 1.4×10^{-12}, and in the L region is 1.65×10^{-12}. The agreement as to order of magnitude is gratifying, and suggests that we are working along the right line. Nevertheless the differences between the theoretical and the experimental values are too great to be accounted for as experimental error, showing that our assumptions must in some way be modified.

The only considerable assumption that has been made in deriving equation (33) is that slightly damped electronic oscillators exist of natural frequencies distributed between the critical absorption frequency and infinity. It must be confessed that the assumption does not appear very plausible. Nevertheless, the fact that without introducing the quantum concept we can thus obtain a formula which is in many ways so satisfactory, strongly suggests the X-ray absorption is not a quantum phenomenon. The results support rather the view that X-ray absorption is a continuous process, obeying the usual laws of electrodynamics.

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[1] N_K, N_L etc. are given by atomic theory as 2, 8, etc., the exponent of λ is determined by wave-length measurements, and the other quantities have fixed, known values.

Quantum theory of X-ray absorption.—Since the above discussion was written, there has appeared a theory of X-ray absorption by L. de Broglie which, although based upon widely different assumptions, leads to nearly the same result as that expressed by equation (33). On the assumptions, 1. that Wien's energy distribution law holds for black body radiation for X-ray frequencies, 2. that Kirchhoff's law relating the emission and absorption coefficients of a body is valid, and 3. that Bohr's hypothesis connecting the energy of the electron's stationary states with the corresponding critical absorption frequencies is correct, de Broglie shows that the atomic fluorescent absorption should be[1]

$$\frac{\tau}{\nu}=\frac{1}{8\pi kcT}\cdot\Sigma\eta_p A_{ip}^{\;n} E_p\cdot\lambda^3. \tag{34}$$

Here k is the Boltzmann constant, c the velocity of light, T the absolute temperature, A_{ip}^{n} the probability that an atomionized by the loss of an electron from the p^{th} shell will return to its normal condition in unit time, E_p is the energy required to remove an electron from the p^{th} orbit, and η_p is an undetermined constant whose value is probably not far from unity.

In view of the soundness of his assumptions, the theoretical basis for the λ^3 law of absorption seems very strong. While some of the experiments have seemed to throw doubt on the exact validity of the third power relation, those in the neighborhood of the critical wave-lengths are performed under adverse conditions, and those at short wavelengths are difficult to interpret because of the unknown magnitude of the scattering. In general the experiments afford a satisfactory confirmation of the cube law over a wide range of wave-lengths.

Since the experiments show that the absorption is independent of the temperature and nearly proportional to N^4 or to E_p^2, de Broglie assumes that

$$A_{ip}^{n}=\alpha E_p T=\alpha(\epsilon_n-\epsilon_p)T=\alpha h\nu_p T,$$

where $\epsilon_n-\epsilon_p$ is the energy radiated as the electron returns from the state of p ionization to its normal condition. The constant of proportionality α is evaluated with the help of the principle of correspondence, considering what its value would be for low frequency oscillators in a body at high temperature.[2] He thus finds,

$$\alpha=\frac{8\pi^2}{c^3}\,\frac{e^2}{m}\,\frac{k}{h^2}. \tag{35}$$

The atomic absorption within the K absorption band is accordingly

$$\frac{\tau}{\nu}=\frac{\pi}{c^4}\,\frac{e^2}{m}\,\lambda^3\left[\eta_K N_K\nu_K^2+\eta_L N_L\nu_L^2+\cdots\right],$$

[1] L. de Broglie, Jour. de Phys. et Rad. 3, 33 (1922), cf. also C. R. 171, 1137 (1920).
[2] L. de Broglie, C. R. 173, 1456 (1921).

or substituting the value of D given in equation (32),

$$\frac{\tau}{\nu}=\tfrac{1}{2}D\lambda^3\left\{\left[\eta_K N_K/\lambda_K^2\right]^{\lambda<\lambda_K}+\left[\eta_L N_L/\lambda_L^2\right]^{\lambda<\lambda_L}+\cdots\right\}. \qquad (36)$$

In his papers, de Broglie has assumed that $\eta=1$, which would make this result differ from equation (33) by a factor of ½. But, in a letter to the writer, he has pointed out that his theory does not definitely determine the value of this constant, so that there is no positive contradiction between the two results. It is certainly remarkable that results so nearly identical should be obtained on the basis of wholly different assumptions.

Absorption of very short wave-lengths.—Especial interest attaches to the study of the absorption of the highest frequency radiation, because the modifications necessary to fit this case involve important assumptions concerning the nature of X-rays and the properties of the electron. According to any expression based upon the usual electron theory, the atomic absorption coefficient can never fall below Thomson's theoretical value $N\sigma_0$ for the total scattering. Experimentally, however, it falls to a considerably lower value. For example, in the case of carbon $N\sigma_0=4.0\times10^{-24}$; whereas for 0.1 Å.U., $\mu/\nu=2.8\times10^{-24}$ and for hard gamma rays ($\lambda=0.025$ Å) $\mu/\nu=1.5\times10^{-24}$, less than half of the theoretical value of the scattering alone. The most obvious explanation of this low value of the absorption is that the electron is large enough for destructive interference to occur between the rays scattered by its different parts. Thus we are led to the same hypothesis of an electron of appreciable size that was found necessary to explain the direct experiments on the scattering of X-rays.[1]

In the above discussion (pp 4-11) it was found necessary to take into account the grouping of the electrons within the atom and the size of the electron itself in order to express adequately the scattering of X-rays. Consequently the quantity $N\sigma_0$ which assumes the electrons to act as point charges independent of each other, must be replaced by a term $\underline{\Omega N\sigma_0}$ where the coefficient Ω is defined by the expression for the total energy scattered by an atom, thus:[2]

$$\frac{\sigma}{\nu}=\int_0^\pi\frac{I}{I}\theta\cdot2\pi L^2\sin\theta d\theta\equiv\Omega N\int_0^\pi\frac{I_0}{I}2\pi L^2\sin\theta d\theta\equiv\Omega N\sigma_0. \qquad (37)$$

Here I is the intensity of the incident beam which strikes the atom, I_θ is the intensity of the beam scattered by the atom to a distance L at an angle θ with the incident beam (cf. equations 2 and 3), and I_0 is the corresponding intensity scattered by a point charge electron

[1] It appears possible that the hypothesis of scattering one quantum by each electron may offer an alternative explanation of this reduced scattering. But no quantitative calculations along this line have as yet been made.

[2] A. H. Compton, Phys. Rev. 14, 250 (1919) and Washington University Studies, 8, 107 (1921).

(equation 1). Thus σ/ν can be determined experimentally by integrating the observed scattered intensity over the surface of a sphere. As thus measured, for wave-lengths less than 0.1 Å.U. the atomic scattering rapidly diminishes, until for hard gamma-rays it is only a small fraction of $N\sigma_0$.

A third type of absorption.—There is, however, another type of absorption which becomes prominent at very short wave-lengths. Ishino has shown[1] that the total absorption coefficient of hard gamma-rays, except for the very heavy elements in which appreciable characteristic radiation is excited, is proportional to the atomic number of the absorber. It might be thought that this absorption is due to scattering of the gamma rays; but Ishino and others have shown that only about half of the absorbed energy reappears as secondary rays. The writer has shown that even of these secondary rays only a very small part is of the same wave-length as the primary γ-rays.[2] That a large part of the energy spent by γ-rays is truly absorbed is confirmed by the fact that the number of high-speed secondary β-rays excited in the absorber is also closely proportional to the atomic number, and that they possess energy which is at least a considerable part of the absorbed energy.[3] Thus in the case of gamma-rays there exists a true absorption which is per atom proportional to the atomic number.

The remarkable simplicity of Owen's $N^4\lambda^3$ relation for fluorescent absorption of X-rays suggests strongly that the energy thus taken into account is dissipated by a single process, directly connected with the excitation of the characteristic fluorescent radiations. The true absorption of gamma-rays, which is proportional to the first power of N and in which the characteristic radiations are not concerned, is therefore presumably due to a different process.

Two alternative hypotheses of the origin of this true absorption of γ-rays may be presented. 1. We may think of this energy as spent in exciting a form of general fluorescent radiation through the medium of the secondary β-rays. Or 2. we may suppose, on the quantum idea of scattering, that when an electron receives a quantum of γ-ray energy, the momentum of the quantum gives to the scattering electron a considerable momentum, so that the scattered energy is equal to the energy absorbed less the kinetic energy of recoil of the scattering electron. On either view, the secondary rays will possess only a fraction of the energy removed from the primary beam, the remaining energy appearing as kinetic energy of the secondary photoelectrons or β-rays.

In any case, the difference between the energy removed from the γ-ray beam and the total energy in the secondary rays is a type of

[1] M. Ishino, Phil. Mag. 33, 140 (1917).
[2] Cf. Supra, p. 7.
[3] Cf. Supra, p. 29.

truly absorbed energy. We may call this the "momentum" absorption, since it seems to be due to the momentum carried by a quantum of radiation. The corresponding "momentum absorption coefficient" may be designated by the letter ω. This quantity is similar to the fluorescent absorption coefficient τ in that they both represent true absorption, and is similar to the scattering coefficient σ in that both are nearly proportional to the atomic number, and are probably both associated with the scattering process.

While the existence of this momentum absorption has been experimentally established only in the case of γ-rays, there is no reason to doubt its existence also in the case of X-rays. Indeed, the observation that the secondary X-rays are of longer wave-length than the primary (supra p. 17) would seem to require its existence. Its magnitude is presumably a function of the wave-length which increases rapidly with the frequency as the frequency of γ-rays is approached. It appears probable that it should again decrease for still higher frequencies. Thus we may write for the atomic momentum absorption

$$\frac{\omega}{\nu} = R(\lambda).\ N, \qquad (38)$$

where R (λ) is probably always less than σ_0.

The general formula for the absorption of X-rays is accordingly,[1]

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[1] From theoretical considerations, the writer (Phys. Rev. 14, 247, (1919), has suggested a modification of the principal absorption term τ/ν. For if (as assumed above) the energy spent in fluorescent absorption is transferred to the translational vibrations of the absorbing electrons, the absorption will be proportional to the square of the acceleration to which the electron is subject by a given electric field. But when the wave-length is comparable with the electron's diameter, the phase of the wave will differ over the electron's different parts, and the maximum acceleration will be less than for a smaller electron of the same mass. A factor should then be introduced into the fluorescent absorption term which is

$$\varphi = \left\{\frac{\text{acceleration of real electron}}{\text{acceleration of point charge electron}}\right\}^2.$$

The value of this ratio for a spherical shell electron, for example, is

$$\varphi = \sin^2\left(\frac{2\pi a}{\lambda}\right) \Big/ \left(\frac{2\pi a}{\lambda}\right)^2,$$

where a is the radius of this electron. Using the value 2×10^{-10} cm. for a, as determined by scattering measurements, for $\lambda = 0.1$ Å, $\varphi = 0.57$, and for $\lambda = 0.025$ Å, $\varphi = .04$. According to the theoretical formula

$$\tau/\nu = K\varphi N^4\lambda^3,$$

we should therefore expect the values of τ/ν to depart markedly from the λ^3 law at .1 Å.U., and to become almost negligible at 0.025 Å.U. The fact that Richtmyer's and Hewlett's measurements at .1 Å.U. show no such variation might perhaps be due to compensating changes in ω or σ. But if the writer's value of 0.025 is correct for the wave-length of the gamma rays from RaC, the characteristic fluorescent absorption of these rays by lead agrees accurately with the value extrapolated from longer wave-lengths *if φ is unity.*

Since our knowledge of the mechanism of absorption is not as complete as that of scattering, however, this contradiction does not mean that our assumed size of the electron is wrong, but that we don't know as much about the absorption process as we thought we did. It would appear from de Broglie's theory of absorption that the cube law should hold whatever the size of the electron.

$$\left.\begin{aligned}\frac{\mu}{\nu}&=\frac{\tau}{\nu}+\frac{\omega}{\nu}+\frac{\sigma}{\nu},\\&=KN^4\lambda^3+RN+\Omega N\sigma_0.\end{aligned}\right\}\quad(39)$$

Here K, N, λ and σ_0 are defined as in the empirical formula (21). Ω, determined by equation (37), is a function of the wave-length and the size of the electrons, as also presumably is the function R. The first term on the right hand side represents the energy which is used in exciting the characteristic radiations of the absorber, the second accounts for that used in producing the general fluorescent X-rays or in giving momentum to the scattering electrons, and the third indicates the X-ray energy truly scattered by the atom[1].

Conclusions.—While the experiments seem to have established the physical existence of these distinct types of absorption, the theory of X-ray absorption presents some fascinating unsolved problems. Thus, the "momentum absorption" ω has almost escaped detection, and from our present meager knowledge it is not possible to come to a definite conclusion concerning its origin. The absorption due to scattering is apparently in accord with the direct measurements of the scattered rays, and is explicable on the basis of the classical electrodynamics if the radius of the electron is of the order of 4×10^{-10} cm. From an experimental stand-point the principle absorption term $KN^4\lambda^3$ is best established. Its form and order of magnitude are in accord with the classical electrodynamics. The sudden changes in the value of K at certain definite wave-lengths are, on the other hand, apparently determined by quantum conditions. But there is some unknown process of damping the absorbing electrons' oscillations, which transforms radiant energy into the quanta of kinetic energy carried by the photoelectrons. The fact that the usual electron theory goes so far in explaining absorption phenomena, leads one to believe that here may be a vantage point for attacking the problem of the connection between the electron and the energy quantum.

[1] It is worth calling particular attention to the fact that the magnitude of the true absorption term ω *increases* with the frequency. The total absorption probably always diminishes with shorter wave-lengths, since increase in ω is accompanied by decrease in the scattered energy σ. But for light elements and wave-lengths so short that τ is small, the true absorption is due almost wholly to the momentum term ω, and hence (at least until extremely short wave-lengths are reached) increases as the wave-length is shortened.

In much practical work, as for example in the therapeutic use of X-rays, broad beams are employed, so that much of the scattered radiation is present with the transmitted rays. Under such conditions the effective absorption coefficient lies somewhere between the total absorption μ and the true absorption $\tau+\omega$. It appears highly probable that in such cases the increase in ω might for light elements more than balance the decrease in τ and the part of σ lost to the beam, as very short wave-lengths are approached. There should accordingly be an optimum wave-length which, under given conditions, should be more penetrating than longer or shorter waves. It is possible that hard gamma rays, for which ω is about equal to σ, are on the short side of this optimum wave-length. If this is the case, X-rays with a somewhat longer wave would be preferable in therapeutic work because they would have greater penetration.

IV. The Refraction of X-rays

Theory.—According to Lorenz's theory of dispersion[1], if the frequency ν of the radiation transmitted by a substance is high compared with the natural frequency of the electrons in the substance, its index of refraction μ is approximately

$$\mu = 1 - \frac{ne^2}{2\pi m\nu^2}, \tag{40}$$

where n is the number of electrons per unit volume, and e and m have their usual significance. If X-rays of ordinary hardness ($\lambda = 0.5$ Å.U.) traverse a substance of density S, this expression indicates an index of refraction which is less than unity by about 1×10^{-6}. With the softest X-rays that have been examined spectroscopically ($\lambda =$ about 10 Å.U.), the refractive index should, however, differ from unity by as much as 4×10^{-4}. A difference of this magnitude should be measurable.

Webster and Clark[2] have calculated the refraction index in the neighborhood of an X-ray absorption band on the basis of the view, discussed above, that the absorption of X-rays is due to a frictional resistance to the forced oscillations of the electrons. Assuming that the energy thus dissipated is wholly accounted for by the characteristic fluorescent absorption τ, they find that, except near a critical absorption frequency, the index of refraction is expressed by

$$\mu = 1 - \Sigma_K \frac{\tau_K\lambda_K}{4\pi^2}\left\{\frac{\nu_K{}^2}{\nu^2} + \frac{\nu_K{}^4}{\nu^4}\log\left|1 - \frac{\nu^2}{\nu_K{}^2}\right|\right\}. \tag{41}$$

Here λ_K and ν_K are the critical wave-length and frequency respectively of the K absorption band, τ_K is the increase in the linear absorption coefficient as the critical K absorption frequency is passed, and the summation is taken over the absorption limits associated with the K, L, M etc. radiations. Except in the neighborhood of a critical absorption frequency, the effect on the refractive index of the electrons responsible for the characteristic fluorescent absorption as thus calculated is not as great as that due to the remaining electrons, as calculated from equation (40).

At the critical absorption frequency Webster and Clark[143] show that there occurs a maximum refractive index given approximately by

$$\mu = 1 + \frac{\tau_K\lambda_K}{4\pi^2}\log\left(\frac{\nu_K}{\Delta\nu_K}\right), \tag{42}$$

where $\Delta\nu_K$ is the range of frequency through which occurs the sudden rise in absorption near the critical frequency ν_K. In the case of a rhodium prism, $\tau_K\lambda_K/4\pi^2$ is about 10^{-7}. Thus unless $\Delta\nu_K$ is extraor-

[1] H. A. Lorentz, "The Theory of Electrons," 2nd Ed., p. 149.
[2] D. L. Webster and H. Clark, Phys. Rev. 8, 528 (1916).

dinarily small, even here the refractive index will differ only slightly from unity.

Experiments.—In his original examination of the properties of X-rays, Roentgen tried unsuccessfully to obtain refraction by means of prisms of a variety of materials such as ebonite, aluminium and water. Previous to the use of homogeneous rays reflected from crystals, perhaps the experiment conducted under conditions most favorable for measurable refraction was one by Barkla[1]. In this work X-rays of a wave-length which excited strongly the characteristic K-radiation from bromine were passed through a crystal of potassium bromide. The accuracy of his experiment was such that he was able to conclude that the refractive index for a wave-length of 0.5 Å.U. probably differed from unity by less than 5×10^{-6}.

A very satisfactory test of the refraction of homogeneous X-rays has been made by Webster and Clark.[*] They found that the refractive index for the different K lines of rhodium, transmitted by a rhodium prism, differed from unity by less than about 3×10^{-4}. These negative results are in accord with the electron theory of dispersion outlined above.

Variations from Bragg's law.—A direct test of the refraction of very soft X-rays is difficult, because of their strong absorption. It has been observed, however, by Stenström[2] that for wave-lengths greater than about 3 Å.U. reflected from crystals of sugar and gypsum, Bragg's relation

$$n\lambda = 2D \sin\theta$$

does not give accurately the angles of reflection. He interprets the difference as due to an appreciable refraction of the X-rays as they enter the crystal. Stenström's interpretation of his experiments has been criticized by Knipping[3], who tried to explain the discrepancy as due to a particular spatial arrangement of the atoms in crystals; but a more careful analysis by Ewald,[4] has shown that such an hypothesis is inadequate to explain the result. In fact, Ewald's calculations show a quantitative agreement between the discrepancies observed by Stenström and those calculated on the hypothesis of refraction.

The fact that Bragg's law cannot be strictly true seems to have been pointed out first by Darwin,[5] who gives for the difference between the observed glancing angle θ and the angle θ_0 anticipated from Bragg's formula,

$$\theta-\theta_0=(1-\mu)/\sin\theta\cos\theta.$$

[1] C. G. Barkla, Phil. Mag. 31, 257 (1916).
[2] Stenström, dissertation, Lund (1919).
[3] P. Knipping, Zeits. f. Phys. 1, 40 (1920).
[4] P. P. Ewald, Phys. Zeits. 21, 617 (1920).
[5] C. G. Darwin, Phil. Mag. 27, 318 (1914).
[*] See Note 2, page 48.

a more useful expression for determining the index of refraction from these measurements is

$$1-\mu=\frac{\lambda_1-\lambda_2}{\lambda_1}\cdot\frac{n_2^2}{n_2^2-n_1^2}\sin^2\theta_1, \tag{43}$$

where λ_1 and λ_2 are the apparent wave-lengths as measured in the n_1 and n_2 orders respectively. If the index of refraction of the crystal is known, the true wave-length can be calculated from the formula

$$\lambda=2D\frac{\sin\theta}{n}\left(1-\frac{1-\mu}{\sin^2\theta}\right). \tag{44}$$

Duane and Patterson[1] and Siegbahn[2] have noticed that even with ordinary X-rays the wave-lengths observed in different orders do not agree. Thus for the tungsten $L\alpha$ line, $\lambda=1.473$ Å., Duane and Patterson find $\lambda_1-\lambda_2=.00015$, whence $1-\mu$ for the calcite crystal used is $+8\times10^{-6}$. Similar measurements on $\lambda=1.279$ and 1.096 Å. give $1-\mu=10$ and 3×10^{-6} respectively. From equation (40) the corresponding theoretical values of $1-\mu$ are about 9, 7 and 5×10^{-6} respectively. Thus the index of refraction is less than unity and is close to the theoretical value.

Total Reflection of X-rays.—Since the refractive index is less than unity, a beam of X-rays striking a plane surface at a sufficiently large angle of incidence should be totally reflected. The critical glancing angle is given by

$$\cos\theta=\mu$$

or

$$\sin\theta=\sqrt{2}\sqrt{1-\mu}. \tag{45}$$

For $\lambda=1$ Å., the value of $1-\mu$ for a substance of density 3 is given by equation (40) as about 4×10^{-6}, in which case $\theta=9.6$ minutes of arc—a readily measurable deflection.

The writer has tried the experiment of reflecting the tungsten line $\lambda=1.279$ Å from surfaces of glass and of silver coated with lacquer.[3] The results are shown in Figure 13, which shows the intensity of the reflected beam at different angles θ. The theoretical values of the critical angle are calculated by equation (45), using the index of refraction given by Lorentz's formula in its more exact form. In view of the difficulties in measuring these small angles and the uncertainties with regard to the densities of the surfaces, these experiments are in surprising accord with the theory. The experimental values for $\lambda=1.279$ are for crown glass, density 2.52, $1-\mu=5.0\times10^{-6}$, and for a thin silver film on glass, $1-\mu=20.9\times10^{-6}$.

Further experiments showed that within the critical angle the re-

[1] Duane and Patterson, Phys. Rev. 16, 532 (1920).
[2] M. Siegbahn, C. R. 173, 1350 (1921); 174, 745 (1920).
[3] A. H. Compton, paper before American Phys. Soc. Apr. 22, 1922.

flection was indeed specular and nearly total, and that the quantity $1-\mu$ for wave-lengths greater than .5 Å. U. is at least roughly proportional to the square of the wave-length. Thus the experiments on the refraction of X-rays are all in accord with the classical electron theory.

If the number of effective electrons per atom is assumed equal to the atomic number, equation (40) gives a moderately accurate means of

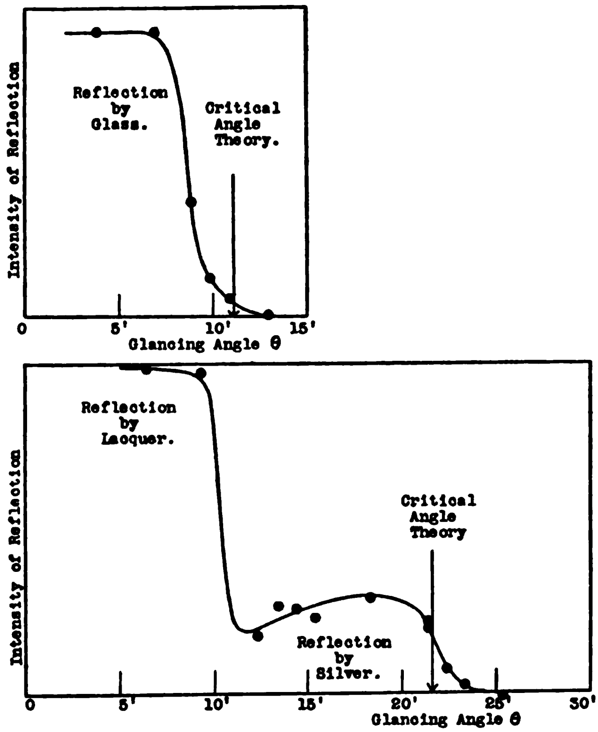

FIGURE 13. The total reflection of wave-length 1.279 Å. from various surfaces, showing the theoretical and experimental values of the critical angle.

measuring the wave-length, which is independent both of any crystalline grating space and of the quantum hypothesis. Or assuming the wave-length to be known, the measurements of the critical angle show that the number of electrons per atom affected by X-rays is probably within 5 per cent of the atomic number. This estimate has a very decided advantage over similar estimates made from X-ray scattering, since the intensity of scattering is affected by the grouping and size as well as the number of the electrons, while the refractive index depends only upon the number of electrons per unit volume (if no appreciable resonance occurs). These X-ray refraction measurements therefore afford a valuable confirmation of our estimates of the wave-length of X-rays and of the number of electrons in the atoms.

V. Some Applications of These Secondary Radiations to Physical Problems

1. That the secondary radiations produced by X-rays constitute a most effective tool for examining the structure of matter is apparent when one recalls that the very definite knowledge we have concerning the number of the electrons in matter and the arrangement of atoms in crystals came first from this source. The basis of the power of this tool lies in the fact that the wave-length of X-rays is comparable with the dimensions of the portions of matter we are studying. It follows that a study of scattered X-rays is capable of giving much the same information about atoms and electrons as may be given about dust particles by a study of the light which they scatter. The information is almost as definite as if we could look into the atom with a super-microscope which, since it employs X-rays or γ-rays instead of light, has a resolving power 10,000 to 100,000 times that of the best optical microscope. In studying the nature of radiation, the secondary X-rays present an equally valuable method, since for these high frequencies the magnitude of an energy quantum is relatively large. It is consequently possible in the X-ray region to distinguish more clearly between those phenomena which involve quantum principles and those which are to be explained on the basis of the classical electrodynamics.

It would be impossible in the compass of such a report as this to discuss adequately the information afforded concerning these major physical problems by the study of secondary X-rays. Nevertheless, certain problems which may be attacked with some success by this method may profitably be mentioned in order to give an idea of what can be accomplished. Since the applications of secondary X-rays to the problem of the structure of matter assume a knowledge of the nature of the X-rays, we shall first discuss the nature of radiation from the standpoint of X-rays.

2. *Nature of Radiation.*—Let us recall that it has been found possible to account satisfactorily for many of the experiments on the scattering of X-rays on the basis of the classical electrodynamics (cf. supra, pp. 4 to 11). The inadequacy of the quantum conception as applied to the scattering of radiation may be shown in the following manner. Experiment shows that a cathode electron at impact may give rise to one quantum of radiant energy of frequency $\nu = Ve/h$, where V is the potential applied to the tube, or to several quanta of lower frequency. Let us consider the case where $\nu > Ve/2h$, so that not more than one quantum of energy of this frequency can be radiated at the impact of each electron. We find that the resulting X-ray when scattered by matter shows the phenomenon of excess scattering. But this phenomenon is to be

accounted for by the fact that the incident X-ray *excites secondary radiation of the same frequency from several electrons in the same atom,* with the result that these radiations cooperate in the forward direction and partially interfere with each other at larger angles with the incident beam.

If, however, it is assumed that each electron both absorbs and emits radiation in quanta, the primary ray can excite radiation from only one electron, and these interference effects become inexplicable. If the radiation is gradually absorbed by all the electrons traversed and if each electron radiates when its energy exceeds the quantum, it is possible that a single quantum in the primary ray might occasionally liberate several quanta of secondary energy (though of course on the average not more than one quantum could be liberated). The chance that these several quanta would all be radiated by electrons in the same atom would, however, be so small as to be wholly inadequate to account for the observed interference effects. The same difficulty arises in explaining the reflection of X-rays by crystals, where in order to account for the appearance of the reflected beam as a sharp line it is necessary to suppose that a large number of electrons in the crystal emit radiation in their proper phases when excited by a single quantum of incident radiation. Thus the interference phenomena occurring in the scattering and reflection of X-rays are inconsistent with the view that an electron always emits scattered radiation in quanta.

There remains the possibility that the radiation itself always occurs in quanta, but that when scattering occurs the quantum of energy is radiated not by a single electron but by some group of electrons, affected by the incident wave. On this view it is a matter of probability whether or not an incident ray shall pass through a scattering body. If it passes through, it remains undiminished as a whole quantum; if it is scattered, some group of electrons in the body are effective, and the whole quantum is scattered, none passing through. This view also requires radiation of energy in discrete quanta, though now the energy quantum may be radiated by any number instead of by a single electron

3. It is important therefore to point out that a quantum of radiant. energy cannot always retain its integrity, and that its parts may be separately scattered and absorbed. Hence scattered energy is not necessarily radiated in quanta, nor is radiation necessarily absorbed in integral quanta.

Perhaps the most convincing example of the division of radiation into parts of less than a quantum is that of the Michelson interferometer. We may suppose that one quantum of energy of wave-length λ strikes the semi-reflecting mirror, half being reflected to the movable mirror and half transmitted to the fixed one. The returning waves recombine at the semi-silvered surface, and in order that the energy quantum may remain complete, we may suppose that both components proceed together, either toward the eye or in the direction of the source. If the mirror is exactly half-silvered, the probability will be equal for the two directions. It is essential, however, that the initial division of the quantum occur at the half mirror, for if all the radiation during any given period went to either the movable or the fixed mirror, there would be nothing with which it could interfere on returning to the half mirror. It seems that the only possible interpretation of this interference is that while part of the energy quantum goes to the movable mirror, another part is simultaneously going to the fixed one. Thus neither part of the divided beam can carry the whole quantum of energy.

It remains to show that such a part of the original quantum can by itself be absorbed or scattered. Let us suppose the silvering of the mirrors is accurate, so that the interference of the rays at the eye is complete. Then imagine a thin absorption screen placed in the path of one of the divided beams. This absorption might for example be due to a slight tarnish on one of the mirrors. That absorption actually occurs is made evident by the incomplete interference of the recombined beam. But we have seen that neither part of the divided light beam possesses a whole quantum of energy. There accordingly appears to be no escape from the conclusion that radiation may be absorbed in amounts less than a quantum.

In a similar manner the light may be reflected or scattered from the surfaces of an interferometer mirror. It seems that in this scattering process we have an example of radiation of energy in smaller units than a quantum. The only escape from this conclusion would be to imagine a gradual absorption of energy by the electrons on the mirror surface, until a whole quantum is accumulated, and then a simultaneous emission of the radiation by the electrons each in its proper phase to produce interference effects. Such a view presents very grave difficulties.

A consideration of these and similar interference phenomena, whose importance must not be minimized, seems to lead with certainty to the conclusion that under certain conditions radiation does not occur in a definite direction, nor in definite quanta; that radiation may be absorbed in fractions of a quantum; and that, in the process of scattering at least, radiation may be emitted in fractions of a quantum. Most of the experiments on X-ray scattering and reflection, on the other hand, receive satisfactory explanation if the X-rays spread over a wide enough solid angle to excite oscillations of a large number of electrons in their proper phases. This study therefore supports the view that radiation occurs in waves spreading throughout space in accord with the usual electrical theory.

The experiments described above (p. 16), showing that the wave-length of the scattered X-ray is greater than that of the incident X-ray, present, however, a serious difficulty to this conclusion. This change in wave-length was found to receive quantitative explanation on the view that the radiation was received and emitted by each scattering electron in discrete quanta. No alternative explanation has as yet suggested itself. Nevertheless, the cogency of the argument based on interference phenomena is so great that it seems to me questionable whether the quantum interpretation of this experiment is the correct one

4. If then radiation may under certain conditions be emitted in infinitesimal fractions of a quantum, and if absorption is a continuous process, the question arises, has the quantum any real physical signi-icance? To this our study of X-rays gives a definitely affirmative answer. Thus we have seen that de Broglie's and Ellis' experiments (supra p. 26) are in accord with the view that each photoelectron leaves its normal position in the atom with a kinetic energy $h\nu$, where ν is the frequency of the incident rays. Conversely, Duane and Hunt's experiments indicate that if the whole kinetic energy of a cathode ray is transformed to radiation at a single impact, then $E_{\text{kinetic}} = h\nu$. A similar relation also expresses quantitatively the frequency of the rays emitted as an electron falls from one energy level to another within the atom. It thus appears that the quantum law may describe a reversible mechanism whereby energy may be interchanged between radiation and the kinetic energy of an electron.

This mechanism is presumably that which is responsible for the fluorescent absorption of X-rays, since it is the energy thus absorbed which appears again as the kinetic energy of the photoelectrons. But the energy dissipated in scattering is not thus transformed, and need not therefore have any dependence upon the quantum mechanism. On this view there is no reason to question the application of the classical electrodynamics to the problem of scattering; so calculations on this basis may be used in studying the structure of matter. Such calculations may be employed with the greater confidence since we have found them capable of explaining the principal phenomena of X-ray scattering.

5. *The Structure of Matter.*—The advances in our knowledge of the structure of matter which have resulted from a study of secondary X-rays can merely be mentioned. Barkla in 1911 estimated the number of electrons in the atom necessary to account for the intensity of X-ray scattering on the basis of Thomson's theory. We have seen (supra, p. 4) that while his method calculation may be questioned, a more critical examination of the problem leads to about the same result: the number of electrons per atom is equal to about half the atomic weight.

Following Laue's discovery of X-ray diffraction, the work of the Braggs' and others has given us definite information concerning the arrangement of the individual atoms in crystals. Besides its value in crystallography, this study has thrown new light on the nature of cohesion, chemical valence, and other problems connected with the solid state. A most valuable method of pursuing this investigation has been developed by Debye and Scherrer,[1] Hull[2] and their co-workers, in which powdered crystals instead of large ones are employed. This method has made possible the study of many substances whose structure could not have been determined by the Bragg method.

The application of these secondary radiations to the study of atomic structure has hardly commenced. Experiments such as those of de Broglie and Ellis on the speed of the photoelectrons ejected by X-rays (supra, p. 26), tell us the energy with which the different electrons are held in the atom. Measurements of absorption coefficients and critical absorption wave-lengths make possible a determination of the number of electrons at each energy level in the atom (cf. supra, p. 41). The writer has shown[3] that a study of the intensity of X-ray spectra is capable of supplying rather definite information concerning the arrangement of the electrons in atoms; and recently Bragg, James and Bosanquet have made a rather extensive investigation from this standpoint of the distribution of the electrons in rock-salt.[4] The intensity of X-ray scattering by amorphous materials should lead to somewhat more definite results concerning this distribution; but such an investigation has not yet been seriously attempted. However, unless some unforeseen difficulty presents itself, it seems safe to predict that such studies will give us within a decade information concerning the arrangement of the electrons in the lighter atoms as definite as our present knowledge of the positions of the atoms in crystals.

It is difficult to overestimate the power of this new tool which is supplied us for the study of the structure of matter. The use of the tool has as yet hardly begun. But the successes already achieved give us hope that our knowledge of the nature of matter may thus be rapidly increased.

---

[1] Debye and Scherrer, Phys. Zeits. 17, 277 (1916) et. al.
[2] A. W. Hull, Phys. Rev. 10, 661 (1917) et. al.
[3] A. H. Compton, Phys. Rev. 9, 29 (1917).
[4] Bragg, James & Bosanquet, Phil. Mag. 41, 309 (1921); 42, 1 (1921).

Vol. 4, Part 3 OCTOBER, 1922 Number 21

BULLETIN

OF THE

NATIONAL RESEARCH COUNCIL

HIGHWAY RESEARCH PROJECTS IN THE UNITED STATES

Results of Census

by

Advisory Board on Highway Research, Division of Engineering, National Research Council, in cooperation with the Bureau of Public Roads, United States Department of Agriculture

by

William Kendrick Hatt, *Director, Advisory Board*

Published by the National Research Council

of

The National Academy of Sciences

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1922

Announcement Concerning Publications
of the
National Research Council

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"What is discovered at the University of Cambridge, of which I happen to be Chancellor, is equally of use in Paris, Chicago, Japan, and at the ends of the earth. It is a gift to mankind."—Lord Balfour, at the inaugural meeting of the British Institute of Physics.

PREFACE

This bulletin contains a census of 479 research projects in highway engineering and highway transport, current or recently completed, in which are sought those data upon which highways may be wisely selected, economically built to conform to the mechanics of the situation, and operated in the interests of the public. It will indicate the sources of current information and the fields of research which at present are not actively occupied, and will aid intercommunication of research workers.

Those who seek completeness of record and logical perfection of analysis of projects will be disappointed. Among the large number of researchers and the great variety or organizations, some projects must have escaped notice. Certain of those to whom questionnaires were addressed have been too busy with pressing duties of production and construction to give them their attention. There has evidently been a misconception in the minds of individuals that the inquiry concerned only those projects in which highway organizations were directly engaged. As a matter of fact, a large part of the technical study of materials of construction is fundamental and useful in highway engineering. Engineers, geologists, educators, city engineers, economists, all contribute.

Industrial laboratories are in some cases naturally not in a position to give publicity to researches directed to the development of special products.

The analytical key of topics (see Part III) is for filing research projects. It contains more than one compartment where a given project might have been placed. For instance, a test of Portland cement concrete might be assigned either to "Aggregates" or to "Portland cement concrete." The decision as to its position will depend upon the purpose of the test—whether for the determination of the value of a deposit, or for the disclosure of a law underlying the proportioning and use of concrete. The cross-references will assist the reader. One difficulty arises in deciding as to the extent to which a project is of purely local significance, as a test of sand or of an aggregate, or a study of traffic. In case of doubt, the project is included.

Some key items appear in several divisions, as, for instance, Maintenance, Design, Construction, and Materials. The project would be listed according to its purpose.

The compilers of this bulletin have used no criterion to separate those projects which have demanded only a routine process of mechanical testing from those that are colored with the qualities of broad research on the higher planes. All reported projects which are likely to be of service to highway officials, and other students of highway problems, have been included.

The questionnaires and the text of the Bulletin purposely make no mention of results of investigations. In this way researchers are spared the embarrassment of premature announcements of conclusions.

Intercommunication between research workers in this field, at present largely lacking, will be furthered by the Bulletin.

While a fixed rule has not been applied to all projects, the plan has been to list only those completed projects that have not been published through channels convenient of access. All projects are arranged alphabetically by states.

A reading of the sections on Economics and on Operation makes it evident that those fields need early occupation by trained research workers because such studies are required not only to determine the investment value of our highways in their yield of economical transportation and other benefits, but also to determine the place which highway transportation should occupy in a coordinated structure of national transportation.

Some projects which were contained in the returns have been listed under Design of Vehicle as related to Road. This field of research, however, has been left to the care of the Society of Automotive Engineers through Dr. H. C. Dickinson, Research Manager. Therefore the analysis of this section in the key list of topics is not complete or developed.

Names of individual researchers have been omitted from the Bulletin; but the questionnaires will be used as a basis for the establishment, in the files of the National Research Council, Research Information Service, of a personnel service bureau in the highway field. This file, which will contain information as to the training and special abilities of workers, will be a division of the general list of researchers in the field of science.

The collection of the data underlying this Bulletin and the subsequent analysis and publication of the projects has only been possible through the cooperation of the Bureau of Public Roads, U. S. Department of Agriculture. Acknowledgment is also due to the assistance of Mr. C. A. Hogentogler, Highway Engineer, U. S. Bureau of Public Roads, and of Miss Agnes W. Ayres and Miss Elise Hatt, Assistants to the Director, Advisory Board on Highway Research.

Suggestions for improvements of a future issue of this Bulletin or one similar will be welcomed by the Advisory Board.

This Bulletin will be followed by a general bulletin in which the status of research in the several fields will be summarized by the Chairmen of the Research Committees of the Advisory Board.

Vol. 4. Part 3 October, 1922 Number 21

BULLETIN

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HIGHWAY RESEARCH PROJECTS IN THE UNITED STATES

Results of Census

BY

ADVISORY BOARD ON HIGHWAY RESEARCH, DIVISION OF ENGINEERING, NATIONAL RESEARCH COUNCIL, IN COOPERATION WITH THE BUREAU OF PUBLIC ROADS, UNITED STATES DEPARTMENT OF AGRICULTURE

BY

WILLIAM KENDRICK HATT, *Director, Advisory Board*

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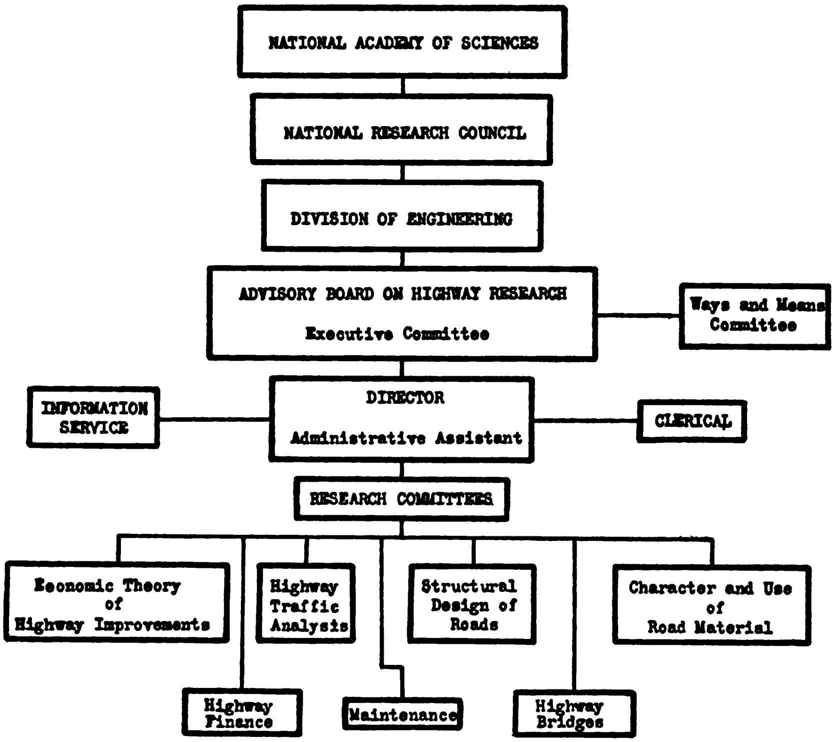

THE ADVISORY BOARD ON HIGHWAY RESEARCH

The Advisory Board on Highway Research of the Division of Engineering, National Research Council, was organized November 11, 1920, for the purpose of preparing a comprehensive national program for highway research; to assist existing organizations to coordinate their activities therein; and to collect and distribute information of completed and current research.

The Advisory Board is composed of the following constituent members:

American Association of State Highway Officials
American Concrete Institute
American Institute of Consulting Engineers
American Road Builders Association
American Society of Civil Engineers
American Society of Mechanical Engineers

American Society for Municipal Improvement
American Society for Testing Materials
Associated General Contractors of America
Association of American State Geologists
Bureau of Public Roads, U. S. Dept. of Agriculture
Bureau of Standards, U. S. Dept. of [redacted] Commerce
Corps of Engineers, U. S. Army
Engineering Foundation
Eno Foundation
National Automobile Chamber of Commerce
National Highway Traffic Association
National Safety Council
Rubber Association of America
Society of Automotive Engineers
Western Society of Engineers

The Research Committees of the Advisory Board are as follows:

(1) COMMITTEE ON ECONOMIC THEORY OF HIGHWAY IMPROVEMENT.

Problems: To determine all of the elements of cost of highway improvement.

The following will suggest the type of research involved:

A. Effect of grades, alignment, rise and fall, weather and speed and methods of operation on cost of transport.
B. Determination of all of the elements entering into the resistance to translation of vehicles (tractive resistance) and magnitude of each element.
C. Determination of the elements of cost of vehicle transportation classed as capital costs, and operating costs exclusive of those included in A and B.
D. To determine the relation between traffic and capital and maintenance costs of roads.

Membership:

Chairman, Prof. T. R. Agg, Iowa State College
Prof. L. E. Conrad, Kansas State Agricultural College
Dr. H. C. Dickinson, Society of Automotive Engineers
Mr. H. S. Fairbank, U. S. Bureau of Public Roads
Mr. A. B. Fletcher, California State Highway Engineer
Prof. H. J. Hughes, Harvard University
Major Mark L. Ireland, Quartermaster Corps, U. S. Army
Mr. E. W. James, U. S. Bureau Public Roads
Mr. H. J. Kuelling, Wisconsin State Highway Commission
Prof. E. H. Lockwood, Yale University
Mr. Charles M. Manly, New York City
Mr. Tom Snyder, Indianapolis, Ind.

(2) Committee on Structural Design of Roads

Problems: To establish all of the data required for the rational design of a road surface.

The following will suggest the type of research involved:

A. To determine all facts relative to the behavior of the soil upon which roads are constructed, when under load from the road structure.
B. To determine the relation between traffic loads and stress in road surfaces, and to establish the laws that control.
C. To determine the effects of the elements on road structures.
D. To determine the structural strength of all types of road surfaces.
E. Relation of the vehicle to the road.

Membership:

Chairman, Mr. A. T. Goldbeck, U. S. Bureau of Public Roads
Mr. C. A. Hogentogler, U. S. Bureau of Public Roads, Secretary
Mr. Lloyd Aldrich, San Francisco, California
Prof. H. C. Berry, University of Pennsylvania
Prof. H. E. Breed, New York University
Prof. F. H. Eno, Ohio State University
Mr. E. W. James, U. S. Bureau of Public Roads
Mr. Clifford Older, Division of Highways, State of Illinois.
Mr. H. G. Shirley, Virginia State Highway Commission
Mr. E. B. Smith, U. S. Bureau of Public Roads
Prof. C. M. Strahan, University of Georgia
Mr. E. W. Templin, Akron, O.
Mr. C. M. Upham, N. C. State Highway Commission
Dr. H. M. Westergaard, University of Illinois

(3) Committee on Character and Use of Road Materials.

Problems: To determine the most effective combinations of materials to give desired strength, and to investigate possible new combinations of materials.

The following will suggest the type of research involved:

A. To establish the most effective combinations of materials now in use, with particular reference to the exigencies of field control.
B. To promote research looking to the establishment of new combinations of materials, or of the new materials suitable for road surfacing.

Membership:
Chairman, Mr. H. S. Mattimore, Pennsylvania State Highway Department
Mr. B. A. Anderton, U. S. Bureau of Public Roads
Mr. E. W. Crum, Iowa State Highway Commission
Prof. F. C. Lang, University of Minnesota
Mr. C. S. Reeve, New York City
Prof. H. H. Scofield, Cornell University
Prof. M. O. Withey, University of Wisconsin

(4) COMMITTEE ON HIGHWAY TRAFFIC ANALYSIS.

Problem: To establish an adequate method of studying highway traffic and to show how traffic records should be interpreted.

The following will suggest the type of research involved:

A. A study of relation of community development to the origin and destination of traffic, and to devise means for estimating potential traffic.
B. Proper method for studying and recording volume of traffic, and for interpreting traffic records.
C. To devise units of measure to apply to traffic and to define those units.
D. To study the relation of highway betterment to traffic increases.

Membership:
Chairman, Mr. Geo. E. Hamlin, Connecticut State Highway Commission
Prof. T. R. Agg, Iowa State College
Prof. A. H. Blanchard, University of Michigan
John N. Cole, Massachusetts State Department of Public Works*
Prof. N. W. Dougherty, University of Tennessee
Dean A. N. Johnson, University of Maryland
Mr. Nelson P. Lewis, New York City
Mr. J. H. Mullen, Minnesota Highway Department

(5) COMMITTEE ON HIGHWAY BRIDGES.

Problem: To establish all of the data required for the design of highway bridges.

The following will indicate the type of research required:

A. Determination of loads for which bridges should be designed.
B. Study impact on highway bridges.
C. Determination of the proper requirements for standards of design as regards allowable stresses, widths and requirements of foundations.

(Committee not yet organized)

*Deceased.

(6) COMMITTEE ON HIGHWAY FINANCE.

Problem: To determine the equitable basis for financing highway improvements.

The following will indicate the type of research involved:

A. Definition of an equitable basis for highway financing.
B. Study possible methods of financing improvements in the various political units.
C. Administrative methods required to insure equitable financing.

Membership:

Chairman, Dr. J. G. McKay, U. S. Bureau of Public Roads
(Committee not yet organized)

(7) COMMITTEE ON MAINTENANCE.

Problem: To determine the relation between traffic and maintenance costs of roads, to investigate methods of maintenance and organization of maintenance forces.

The following will suggest the type of research involved:

A. Establishment of accounting methods that will give accurate data on cost of maintenance.
B. Methods of correlating maintenance costs and volume of traffic.
C. Relation of maintenance costs to methods of maintenance.

Membership:

Chairman, Mr. W. H. Root, Iowa State Highway Department
(Committee not yet organized)

The Advisory Board on Highway Research, Division of Engineering, National Research Council, is not a directing, but a service organization, devoted to the interest of the individual researcher who instinctively resists interference from overhead, and properly guards with care the products of his zealous research. Certain services, however, that can be and that have been rendered the individual researcher are: by soliciting support for his projects, by putting him in touch with his fellow workers through a Census of Research, and by illuminating a particular field with side-lights from allied fields in which a project has its setting. The taxpayer also benefits—through a coordinated research activity on problems selected from a comprehensive program with reference to applicability to present-day problems. The following are examples:*

Highway researches in the field of economics of operation are conducted under Committee No. 1 on the Economic Theory of Highway Improvement, Professor T. R. Agg, Chairman. In New England the

* This and following paragraphs are reproduced from a paper by the author of this Bulletin as published in Proc. Am. Soc. C. E., 1922.

tractive resistance of automobiles and trucks is being measured on several types of roads at speeds up to 30 miles per hour. The vehicles are tested for internal absorption of energy at the Yale laboratory by Professor E. H. Lockwood, and the gross tractive resistance determined on the roads by Major Mark L. Ireland, Q. M. C., who is stationed at the Massachusetts Institute of Technology. Cooperating in this work are:

- Bureau of Public Roads, U. S. Department of Agriculture
- United States Army, Q. M. C. Department
- Committee on Economic Theory of Highway Improvement, National Research Council
- Connecticut Highway Commission
- Department of Public Works, Commonwealth of Massachusetts
- Harvard University
- Massachusetts Institute of Technology
- Yale University
- Society of Automotive Engineers

The investigation includes the effect upon tractive resistance of the temperature of the differential of the vehicle, and of various types of tires. Of course the gross tractive resistance is composed of several elements, such as engine, gearing, inelastic resistance of body, tire, windage, road surface, which distinguishes rolling, impact and displacement resistance. Account must be taken, too, of the absorption and giving out of energy in the rotating parts, in accelerating or retarding speed. The technique of these measurements is not settled. The measuring instruments may be an accelerometer used in coasting tests, or a dynamometer, in towing tests, each of which has its peculiar advantages. The first task is to determine the best instrument for measuring the several elements into which the problem is analyzed. The Bureau of Standards has assembled apparatus of a remarkable degree of completeness and perfection, which automatically records the measurements of some fourteen elements of car performance in road tests.

Professor L. E. Conrad, at Manhattan, Kansas, is conducting a parallel investigation of the resistance of the air to the passage of vehicles.

At the University of Michigan, Professor Lay in a study of economic grades has towed four Packard trucks over gravel and concrete roads, after he had determined their characteristics in the laboratory. The gravel roads varied in condition from wind-swept hard surfaces to a loose gravel surface. The concrete road varied from a very smooth finish to a more or less wavy condition. The gasoline consumption was also reported. The trucks were 5-ton with 3-ton pay load, and were run at a standard engine speed of 1,000 R. P. M., the gears being shifted to meet road conditions. Professor Lay finds that for these trucks, the tractive resistance, including rolling resistance, tire resistance and windage, varied from 40 to 44 pounds per ton on the gravel roads, and from 22 to 26 pounds per ton on the concrete roads, all at a speed of 10 miles per hour. Professor Lay's report, shortly to be published, is a notable contribution.

At Ames, Iowa, Professor T. R. Agg has been investigating for a long time the economics of highway location and operation, especially with a view to determining economic grades. His report is expected to be published by January, 1923.

The end in view is a sound economic theory of highway location and improvement.

Now, each of these investigations forms part of a whole. The individual researchers profit by intercommunication, and in the end the entire results are more likely to be useful in the formation of principles because of a coordinating service.

Again, in the field of design we may instance the experimental road at Pittsburg, California; the Bates test road near Springfield, Illinois; investigations of Mr. Goldbeck, of the Bureau of Roads; and the service test roads at Byberry, Pa., Lancaster, Pa., Casper, Wyoming, Alexandria County, Va., Wilmington, Del., and Milwaukee, Wis., all of which are directed to the same end and will yield a body of coordinated data flowing through the channel of the Committee on Design, and will place the mechanics of slab design on a scientific foundation.

Again, in the field of highway transport, the pioneer transport surveys on the Connecticut roads, inaugurated by Commissioner Charles J. Bennett, have crystallized formerly vague discussions, and have stimulated further work. The examinations by Dean A. N. Johnson of the University of Maryland, by Professor N. W. Dougherty of the University of Tennessee, by Professor T. R. Agg in Iowa, by the Bureau of Public Roads in California, and less complete traffic counts in scores of instances—all of these need to be reported and examined for fundamental laws of traffic by a committee on Traffic Analysis.

A useful and money-saving activity exists in the Research Information Service of the Advisory Board, by which an individual state is informed of the findings from investigations in another state. Cases can be cited in which one state has been saved the delay and cost of an investigation relating to the use of materials, when informed by the Research Information Service of the completed investigation available from the work of another state.

It must be said, however, that duplication is not necessarily an evil; indeed it is useful in fundamental matters, as in the case of the investigation of fatigue of concrete. The parallel investigations at Springfield, Illinois, Purdue University and the University of Maryland, with different methods of attack, will more certainly lead to the truth. However, even in these cases each will profit by communication with the other.

Mr. H. S. Mattimore, Engineer of Tests, State Highway Department, Harrisburg, Pennsylvania, is Chairman of the Advisory Board Committee on the Character and Use of Road Materials, also Chairman of the American Association of State Highway Officials' Committee on Tests and Investigations. The latter committee is divided into subcommittees, each representing a region of the United States. Mr. Mattimore is furnished by the Advisory Board with an administrative assistant, and this combined attack on problems relating to the materials of construction represents a well-organized committee activity.

These few instances will serve, perhaps, to demonstrate the value of the services of the Advisory Board in tying together individual researches.

PART II

METHOD OF PROCEDURE IN HIGHWAY RESEARCH CENSUS

The Highway Research Census was conducted in the following manner:

Questionnaires were sent to six classes of organizations: educational institutions, state highway departments, geological surveys, industrial organizations, cities and counties. The personnel of the educational institutions circularized was taken from the National Research Council files, and certain titles of projects were furnished by the bulletins of the Association of Land Grant Colleges. The personnel of state highway departments was derived from the Highways Green Book and from the U. S. Bureau of Public Roads. Names of various state geologists were supplied by Dr. Homer P. Little, the Executive Secretary of the Division of Geology and Geography in the National Research Council. A list of certain cities over 50,000 in population was taken from the World Almanac. The titles of certain industrial laboratories in the United States were derived from Bulletin No. 16 of the National Research Council lists. Counties active in highway research have been named at various intervals by the U. S. Bureau of Public Roads, although this list has just begun to be compiled.

After the organization record was somewhat complete, a questionnaire for titles of projects and an explanatory letter were printed and sent out (see Blank *a*). When this was returned, a second questionnaire (see Blank *b* and letter) was sent, one sheet for each project named by the researcher. Then, in case no answer was received, from one to three follow-up letters were sent out.

As soon as questionnaire *a* was returned, the information was copied on cards and a geographical file was made. The separate projects and details, derived from questionnaire *b*, were filed according to the *Index of Highway Research* (see page 14).

LETTER A

ADVISORY BOARD ON HIGHWAY RESEARCH

UNDER AUSPICES OF

NATIONAL RESEARCH COUNCIL, DIVISION OF ENGINEERING, AND ENGINEERING FOUNDATION

Office of Director, National Research Council Building

1701 Massachusetts Avenue, Washington, D. C.

DEAR SIR:

The first census of engineering research as secured from replies to the circular letter of March 1, 1920, sent out by the Research Information Service of the National Research Council, gave ample evidence of the existence of data that are of value to the highway engineer, automotive engineer, and highway administrator.

The Advisory board on Highway Research of the Division of Engineering, National Research Council, in cooperation with the United States Bureau of Public Roads, now proposes to compile records of the highway researches of the various research agencies of the United States, and to act as a central distributing agency from which information furnished by highway investigation can be obtained.

For this purpose it is asked that you state briefly on the enclosed forms the titles of: (1) Investigations contemplated or in progress; (2) investigations which have been carried to completion or upon which reports have been issued; and (3) subjects upon which research data are desired.

We have inserted on these forms the information at present on record in our files, and will ask you kindly to correct this if in error.

The early return of the desired information to this office will greatly facilitate the work of compilation. With heartiest appreciation of your cooperation,

Yours respectfully,

W. K. Hatt,

Director.

LETTER B

ADVISORY BOARD ON HIGHWAY RESEARCH

UNDER AUSPICES OF

National Research Council, Division of Engineering, and Engineering Foundation

Office of Director, National Research Council Building

1701 Massachusetts Avenue, Washington, D. C.

Dear Sir:

We greatly appreciate the attention given to the blank forms of the Highway Research Census which were distributed in April by the Advisory Board on Highway Research of the National Research Council.

In order to complete the census, it is necessary to ask you to fill out the enclosed detail blanks upon which we have inserted the information obtained from your recent report which is now on record in our files. We would suggest that you also kindly scrutinize the general form once more for any errors in personnel or address. It is important that this be returned to us along with the information on the detail blanks.

The hearty response to the questionnaire and the large amount of valuable research data brought to our attention, has justified the project. A report of this census will be published in the future and will be sent to your address.

The early return of the desired information to this office will greatly facilitiate the work of compilation.

With heartiest appreciation of your cooperation,

Very truly yours,

W. K. Hatt,

Director.

BLANK A

HIGHWAY RESEARCH CENSUS

ADVISORY BOARD ON HIGHWAY RESEARCH, NATIONAL RESEARCH COUNCIL

IN COOPERATION WITH THE

U. S. BUREAU OF PUBLIC ROADS

NAME OF RESEARCH ORGANIZATION..........

ADDRESS..........

OFFICIAL IN CHARGE..........

TITLE OF POSITION..........

INVESTIGATORS

.... : ....

.... : ....

.... : ....

.... : ....

.... : ....

(1) INVESTIGATIONS CONTEMPLATED OR IN PROGRESS

(LIST BY TITLES)

....

(2) INVESTIGATIONS COMPLETED

TITLE WHEN AND WHERE REPORTED

....

(3) ADDITIONAL SUBJECTS ON WHICH RESEARCH DATA ARE NEEDED BY REPORTER

....

BLANK B

ORGANIZATION.............. **HIGHWAY RESEARCH CENSUS** TEST NO...............

ADVISORY BOARD ON HIGHWAY RESEARCH, NATIONAL RESEARCH COUNCIL

IN COOPERATION WITH THE

U. S. BUREAU OF PUBLIC ROADS

NAME OF ORGANIZATION..........

TITLE OF INVESTIGATION..........

....

OBJECT ....

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APPARATUS..........

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PROCEDURE (ESSENTIAL ELEMENTS OF METHOD, IF NEW)..........

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ARE COPIES OF WORKING PLANS AVAILABLE?..........

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DATE OF BEGINNING..........DATE OF COMPLETION..........

PERSONNEL: INVESTIGATOR IN CHARGE..........

ADDITIONAL INVESTIGATORS	PER CENT TIME	ASSISTANTS	PER CENT TIME
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COOPERATING AGENCIES..........

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IS ADDITIONAL COOPERATION DESIRED?.......... IS THERE SPECIAL NEED OF ASSISTANCE, APPARATUS, OR MATERIALS, CONTACT WITH OTHER INVESTIGATORS, LIBRARY FACILITIES, ETC.?..........

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CHECK OFF FIELD IN WHICH INVESTIGATION LIES..........

ECONOMICS - OPERATION - DESIGN OF ROAD - DESIGN OF VEHICLE - CONSTRUCTION - MATERIALS

....

H-C-2

PART III

DETAILED OUTLINE OF THE FIELDS OF HIGHWAY RESEARCH

10 ECONOMICS

11 *Traffic Studies*

.1 Distribution in region
.14 Seasonal variations
.2 Character
.21 Vehicle
.22 Weight and distribution
.23 Speed
.24 Tire
.25 Commodity, unit loads
.26 Length of haul
.27 Body width and weight
.3 Method of expressing unit of traffic
.4 Predicted changes
.5 Relation to other methods of transport
.6 Central sources of traffic
.7 Commodity movement, volume and type (see also 11.2)
.8 City traffic counts and investigations

12 *Community Needs*

.1 Traffic counts
.2 Selection and location as influenced by localitv
.3 Intangibles

13 *Cost of Transport*

.1 Road
.11 Capital investment
.111 Right of way
.112 Grading
.113 Structures
.114 Surface
.115 Depreciation
.12 Operation
.121 Administration
.122 Maintenance
.2 Vehicle
.21 Capital investment
.211 Depreciation
.22 Operation
.221 Administration
.222 Operating expenses
.3 Equivalent vehicle units in terms of road cost
.4 Economic life of road

14 *Economics of Location*

.1 Cost of distance
.2 Rise and fall
.3 Ruling grade
.4 Curvature
.5 Ruling curve
.6 Tractive resistance
.7 Economic grades
.8 Details of location

15 *Theory of Finance*

.1 Construction and maintenance cost
.2 Benefits received
.21 Property increment of value
.211 Increment of land value
.2111 Dirt roads
.2112 Gravel and macadam roads
.2113 Paved roads
.22 Value to vehicle users
.23 Community value
.3 Sources of revenue
.31 Bond issues
.311 Types and costs
.32 Taxation
.321 Real property
.322 Personal property
.323 Income tax
.324 Inheritance tax
.33 Vehicle fees
.34 Gasoline tax
.35 Miscellaneous sources of revenue

16 *Bibliographies*

20 OPERATION

21 *Control of Traffic*

.1 Routing
.2 Terminals
.3 Franchises
.4 Police regulations
.5 City traffic signs and signals

22 *Accident Insurance*

23 *Planning Systems of Transport*

.1 Financing
.2 Community needs
.3 Relation to other transport organization
.4 Terminals

24 *Maintenance* (see also 35)

.1 Systems (Control and Inspection)
.2 Methods
.21 Joint and crack filling (see also 35.61)
.211 Materials and storage
.212 Equipment
.22 Surface treatments
.221 Materials and storage
.2211 Oiling
.2212 Dust preventives
.222 Equipment
.23 Patching
.231 Materials and storage
.232 Equipment
.24 Reshaping
.241 Machinery and equipment
.3 Snow removal
.4 Replacement

25 *Route Marking*

26 *Cost Accounting Systems*

27 *Safety* (see also 21)

28 *Surface Treatment*

29 *Materials and Storage*

30 DESIGN (road)

31 *Subsoil Studies* (see also 37)

.1 Properties
.2 Bearing power and volume change
.21 Moisture content
.22 Special treatment
.23 Surface deformations
.24 Climatic changes
.3 Distribution of pressure

32 *Drainage, Design of*

.1 Drains
.2 Bridges (see also 36.1)
.3 Culverts (see also 36.2)

33 *Subbase Course*

34 *Base Course* (see also 37, 38, 60)

.1 Concrete
.11 Materials
.12 Proportioning
.13 Thickness
.14 Cross section

.2 Broken stone, gravel, etc.
.21 Thickness
.22 Materials
.23 Proportioning
.24 Cross section
.3 Bituminous concrete or macadam
.31 Materials
.32 Proportioning
.33 Thickness
.34 Cross section

35 *Top Course* (see also 37, 38, 60)

.1 Concrete
.11 Materials
.12 Proportioning
.13 Thickness
.14 Cross section
.15 Type of slab
.151 Solid
.152 Precast (see also 37.5)
.153 Cellular
.16 Joints
.161 Fillers (see also 24.21)
.162 Load spacing
.17 Resistance to traffic and climate (see also 37)
.171 Surface wear
.172 Dusting
.173 Cracking
.174 Loads
.1741 Static (see also 38.1)
.1742 Impact (see also 38.2)
.175 Contraction and expansion
.176 Warping
.18 Distribution of load to subgrade (see also 31)
.19 Effect upon traffic
.191 Tractive resistance (see also 14.6)
.192 Wear of tire
.193 Wear of vehicle
.194 Reinforcing (see also 37)
.195 Theory of
.196 Amount
.197 Kind
.198 Distribution
.199 Direction
.2 Brick and block
.21 Materials
.22 Size
.23 Cushion
.24 Filler
.25 Cross section
.26 Resistance to traffic and climate

.261 Surface wear
.262 Cracks
.263 Contraction and expansion
.264 Warping
.265 Loads
.2651 Static
.2652 Impact
.27 Distribution of load to subgrade (see also 31)
.28 Effect on traffic
.281 Tractive resistance (see also 14.6)
.282 Wear of tire
.283 Wear of vehicle
.29 Joints
.3 Bituminous concrete, bituminous macadam, and sheet asphalt
.31 Materials
.32 Proportioning
.33 Thickness
.34 Cross section
.35 Resistance to traffic and climate (see also 24)
.351 Surface wear
.352 Rutting and raveling
.353 Disintegrating
.354 Loads
.3541 Static
.3542 Impact
.355 Shoving and waving (see also 37.6)
.356 Bleeding
.36 Distribution of load to subgrade (see also 31)
.37 Effect upon traffic
.371 Tractive resistance (see also 14.6)
.372 Wear of tire
.373 Wear of vehicle
.4 Water bound macadam and gravel
.41 Materials and binders
.42 Proportioning
.43 Thickness
.44 Cross section
.45 Resistance to traffic and climate (see also 24)
.451 Surface wear
.452 Rutting and raveling
.453 Disintegrating
.454 Loads
.4541 Static
.4542 Impact
.455 Shoving and waving
.46 Distribution of load to subgrade (see also 31)
.47 Effect upon traffic
.471 Tractive resistance (see also 14.6)
.472 Wear of tire
.473 Wear of vehicle
.48 Dust preventives
.49 Oiling

.5 Top-soil and sand-clay
.51 Materials
.52 Proportioning
.53 Cross section
.54 Dust preventives
.55 Oiling
.6 Dirt roads
.61 Materials
.62 Cross section
.63 Dust preventives
.64 Oiling

36 *Structures*

.1 Bridges (see also 32.2)
.2 Culverts (see also 32.3)

37 *Experimental Roads*

.1 Pittsburg, California
.2 Newcastle County, Wilmington, Delaware
.3 Bates, Illinois
.4 Warren County, New Jersey
.5 Byberry-Bensalem, Pennsylvania
.6 Lancaster, Pennsylvania
.7 Nueces County, Texas
.8 Alexandria County, Virginia
.81 Arlington, Virginia
.9 Milwaukee, Wisconsin
.10 Casper, Wyoming

38 *Design of Slab*

.1 Mechanical analyses and formulae
.2 Impact tests

39 *Additional Construction*

40 DESIGN (vehicle)—census only partly developed

41 *Design of Vehicle*

.1 Power gear ratio
.2 Braking
.3 Distribution of load (see also 38.2)
.31 Sprung
.32 Unsprung
.33 Impact
.4 Tires
.41 Solid
.42 Pneumatic
.43 Cushion
.44 Metal
.5 Economy of operation and maintenance (see also 13.22)

42 *Surface*

.1 Maintenance and operation of vehicle (see also 13.22)
.11 Tractive effort (see also 14.6)
.12 Fuel, oil, etc.
.13 Wear on tires
.14 Depreciation of vehicle
.2 Loads

43 *Alignment*

.1 Curves (speed)
.2 Grades

44 *Cross Section*

.1 Width
.2 Crown

45 *Safety* (see also 21)

50 CONSTRUCTION (methods and plant; see also 60)

51 *Subgrade*

.1 Clearing and grading
.11 Methods (plant and equipment)
.2 Treatment (see also 31.22)

52 *Type*

.1 Concrete
.11 Materials (storage and handling)
.12 Plant
.13 Mixing (see also 64.2)
.14 Placing
.15 Reinforcing (see also 37)
.16 Finishing
.17 Curing (see also 64.2)
.18 Resurfacing
.2 Brick and block
.21 Materials (storage and handling)
.22 Cushion
.23 Laying
.24 Rolling and culling
.25 Filler
.251 Cement grout
.252 Bituminous
.26 Finishing and curing
.3 Macadam and gravel
.31 Materials (storage and handling)
.32 Placing
.33 Applying water or bituminous binder (see also 24.22)
.34 Finishing
.35 Seasoning

.4 Bituminous macadam and concrete
.41 Materials
.42 Placing
.43 Mixing
.44 Finishing
.5 Earth and sand-clay
.51 Materials
.52 Placing
.53 Finishing
.54 Seasoning

53 *Structures* (for design, see 36)

.1 Bridges
.2 Culverts (see also 32)
.3 Retaining walls
.4 Curbs
.5 Guard rail

54 *Road Inspecting and Testing* (see also 64.221)

.1 Instruments
.2 Cores
.3 Specimens

55 *Cost Accounting*

.1 Standard estimating and cost
.2 Cost factors

56 *Construction Contracts*

.1 Standardization
.2 Cost factors

57 *Construction Economics*

.1 Factors for the investigation of the special project

60 MATERIALS

61 *Aggregates and Tests in Mixtures* (see also 64)

.1 Fine aggregates
.11 Sand
.111 Production
.1111 Sources of supply
.1112 Plant and equipment
.1113 Methods
.1114 Preparation and treatment
.1115 Costs
.1116 Trade practices
.112 Tests
.1121 Sampling
.1122 Preparation and treatment for tests
.1123 Methods of tests
.1124 Standard tests
.1125 Fundamental properties
.113 Specifications
.114 Impurities

.2 Coarse aggregates
.21 Stone (see 61.11–61.114 for form of index)
.22 Gravel (see 61.11–61.114 for form of index)
.23 Slag (see 61.11–61.114 for form of index)
.3 Material survey
.4 Low grade aggregates

62 *Bituminous*

.1 Tar
.11 Production
.111 Sources of supply, etc.
.12 Tests
.121 Sampling, etc.
.13 Specifications
.14 Impurities
.2 Asphalt and oils (see 62.1–62.14 for form of index)

63 *Non-bituminous*

.1 Portland cement (see 61.11–61.114 for form of index)
.2 Brick (see 61.11–61.114 for form of index)
.3 Wood block (see 61.11–61.114 for form of index)
.4 Steel (see 61.11–61.114 for form of index)
.5 Granite block (see 61.11–61.114 for form of index)
.6 Clays (see 61.11–61.114 for form of index)
.7 Paints (see 61.11–61.114 for form of index)

64 *Concrete*

.1 Bituminous concrete and tests of mixtures
.11 Theory of proportioning
.12 Method of tests
.13 Fundamental properties
.131 Tension
.132 Compression
.133 Shear
.134 Bond
.135 Flexure
.136 Fatigue
.137 Volume changes
.14 Destructive agencies
.15 Admixtures
.2 Portland cement concrete and mortar
.21 Theory of proportioning
.22 Method of tests
.221 Field tests and cores
.23 Fundamental properties
.231 Tension
.232 Compression
.233 Shear
.234 Bond
.235 Flexure
.236 Fatigue

.237	Volume change
.238	Wear
.239	Consistency
.24	Destructive agencies
.241	Heat
.242	Seawater
.243	Electrolysis
.244	Alkali
.245	Organic material
.246	Climatic
.25	Admixtures
.251	Waterproofing
.252	Hardeners
.253	Accelerators
.254	Lime

For *Fillers*, see 24.211.

PART IV

PROJECTS IN HIGHWAY RESEARCH CURRENT OR RECENTLY COMPLETED

Projects are arranged alphabetically by states and the District of Columbia is treated as a state. For full official name of institutions and organizations, see Part VII, p. 99, Index of Organizations.

10. ECONOMICS

11. TRAFFIC STUDIES

MICHIGAN. *University of Michigan.* Economic problems in highway transport.

MINNESOTA. *State Highway Department.* Traffic Studies. A study of increase in motor traffic.

At about 140 points on the trunk highway system, during week of August 20-26, 1922. Also at 16 points on July 6, 14, 22, 30; August 7, 15, 30. See Circular G 22-51. Automobiles, state and foreign transportation lines, busses and trucks; trucks general, under and over 2 tons; horse-drawn. For report on previous census, 1916, 1917, 1918, 1919, 1920, see Supplementary Report of Commissioner of Highways for 1920, May 1, 1921. Special report for 1921 in blueprint form.

11.1 Distribution in Region

COLORADO. *State Highway Department.* Traffic census investigation.

To obtain density of traffic along various highways and at various places, together with data for bridge design (projected). For traffic census, see Colorado Highway Bulletin, Oct. and Dec., 1918, and Jan., 1919.

CONNECTICUT. *State Highway Commission.* Photographic air survey of traffic on main line between New Haven and New York.

A study of comparative traffic along different points of the route, together with photographic record, showing dangerous conditions, turns in the road, etc. Airplane, automatic camera, etc. Flight begins at New York State Line, and follows along the Post Road so-called, between that point and city of New Haven. Machine will fly an average height of 500 feet. Date of beginning: May 20, 1922. Cooperating agency: American Air Service Corporation.

CONNECTICUT. *State Highway Commission.* Traffic census of state.

See Engineering News-Record, May 18, 1922.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads,* in connection with a study of the California highway system.

The principal traffic count was for an interval equivalent to one day of 16 consecutive hours from 6 a. m. to 10 p. m., and taken at 103 stations between August 7 and October 14, 1920. Supplementary counts were also taken. To check the positions selected for the 103 stations for the principal one-day count, the California State Highway Commission independently selected 187 proposed traffic stations.

MARYLAND. *University of Maryland.* Traffic analysis.

To present traffic analysis data as a guide for distribution of maintenance funds. The result of four years' traffic count at some 180 stations on the State Highway system of Maryland. A traffic map of Maryland has been completed and the text for accompanying bulletin is ready for publication. For map, see Engineering News-Record, June 8, 1922. Cooperating agencies: U. S. Bureau of Public Roads, State Road Commission of Maryland. See Public Roads, July, 1921.

MICHIGAN. *State Highway Commission.* Traffic Count.

Traffic census taken during month of August, 1922. Michigan Roads and Forests, V. 19, No. 30, Aug. 31, 1922, p. 7.

MISSOURI. *State Highway Department.* Modified traffic census.

TENNESSEE. *University of Tennessee.* Highway economics and highway transport survey in typical counties in Tennessee.

To determine kind and volume of traffic, to classify roads according to service they are giving and may give, to project an improvement program, which can be financed and which will be the most economical, considering local materials, present wealth, probable wealth, and traffic. See Bulletin No. 1, Engineering Experiment Station, University of Tennessee, *Highway Economics and Highway Transport in Typical Counties of Tennessee*, by Prof. N. W. Daugherty, 1922. Cooperating agencies: U. S. Bureau of Public Roads, University of Tennessee, Tennessee Highway Department, and Tennessee Counties.

WASHINGTON. *King County.* Traffic census on main trunk line routes from city limits.

Motorcycles, touring cars, stages, trucks under 2 tons, 2–5 tons, over 5 tons, horse-drawn vehicles. August 23–29, 1921, 6 a.m. to 8 p.m. On Seattle–Tacoma road at maximum periods in 1907, 1909, 1911, 1915.

11.11 Seasonal Variations

CONNECTICUT. *State Highway Commission.* Traffic census on Saybrook-Old Lyme Bridge. (July, 1920-July, 1921.)

To study traffic throughout year. Daily count of various classes of vehicles. Third week in October and third week in May represent average traffic for year. Range is 240% to 10% of average. For analysis, see Engineering News-Record, March 23, 1922. (See also University of Maryland, Public Roads, July, 1921.)

11.2 Character

CONNECTICUT. *State Highway Commission.* Studies of traffic.

To determine character of future construction, width of travel path, need of parallel routes, relation of highway to railroad transportation. Platform scales and measuring apparatus, for determining weight on front and rear tires, tire width, etc. Census blank lists, count of vehicles of various classes, origin and destination, commodity. A fourteen-day census taken on Hartford–Springfield road at State line in August, and on Boston Post Road at the town of Greenwich during October. See Engineering News-Record, January 12-May 18, 1922. A second census made August, 1922.

DISTRICT OF COLUMBIA. *Chamber of Commerce of the United States.* Transportation.

A study of growth of investment costs of all forms of transportation in the United States and relation of highway transportation to transportation in general. See paper by J. Rowland Bibbins, Manager of Department of Transportation and Communication, Chamber of Commerce of the U. S., read before the National Construction Conference, Chicago, April 5, 1922, *Transportation-Progress and Development.*

MASSACHUSETTS. *Department of Public Works.* Traffic census on Massachusetts roads.

To determine character and weight of traffic on ten roads leading out of Boston to industrial centers. 5,200 trucks were stopped. Census shows three-fold increase in traffic on Massachusetts highways in 9 years. Engineering and Contracting, Vol. 51, pp. 7 and 8, January 1, 1919.

11.8 City Traffic Count

CALIFORNIA. *San Francisco Department of Highways and Streets.* City traffic census.

To determine the movement of pedestrian traffic in the downtown and business district; to determine the amount of vehicular traffic, both street car and other types. Standard traffic counters. The pedestrian traffic count is made quarterly with the cooperation of the Building Owners and Managers Association of San Francisco. Date of beginning, 1921. Cooperating agency: San Francisco Police Department Traffic Squad.

ILLINOIS. *Chicago. Department of Highways and Streets.* Zoning of activities, manufacturing, business, residences.

Survey of city; making of map; checking up the use, height, and depreciation of buildings; computing the percentage of unoccupied building lots; recording the width of streets; and compiling other useful data which will form the basis for a comprehensive zoning ordinance. Date of beginning, July 22, 1921. In progress.

PENNSYLVANIA. *Pittsburgh.* Traffic count. Taken by Citizens Committee on City Plan cooperating with city.

Street width as multiple of path of travel. Proposed major street plan based thereon. See Proceedings of Engineers' Society of Western Pennsylvania, January, 1922.

12. COMMUNITY NEEDS

ILLINOIS. *University of Illinois.* Haulage conditions in Illinois.

Object, to improve conditions. Data gathered from field trips and questionnaires.

12.1 Traffic Counts

The following references to traffic counts are taken from a valuable article, "The Traffic Census," by Dean A. N. Johnson of the University of Maryland, published in December, 1920, in Public Roads, a magazine issued by the Bureau of Public Roads, Washington, D. C.

ALABAMA. *Jefferson County (Birmingham).*

Traffic observed on practically all of the roads of the county for two days each week, 14 hours per day for three consecutive months. Nine classifications. To study classes of traffic using roads and to determine the class of surfacing material best adapted for given situations.

COLORADO.

Traffic counts 1917 and 1918 daily during August, 6 a.m. to 8 p.m., covering many of the state highways. See Colorado Highway Bulletin, October and December, 1918, and January, 1919, for map of the state highways and total tonnage carried by each.

DELAWARE.

Traffic census at 26 points over two months at intervals of eight days from 7 a.m. to midnight, in 1917 and in 1920.

IDAHO.

Three traffic counts, one each in March, June, September, 1920, from 7 a.m. to 9 p.m. for nine consecutive days.

ILLINOIS.

Traffic count, begun in 1906, for continuous period of two years at 71 stations. Counts were taken on average of 4 days a month from 6 a.m. to 6 p.m. Also during 1910–1912 at many stations established in 1906–1907. See reports of Illinois Highway Commission for 1906, page 22; for 1907, page 23; and report 1910–1912, page 269.

IOWA.

Traffic counts at 47 stations on Federal-aid projects in 1917; and 1918 at 87 tations, 40 of which were located on Federal-aid projects. Counts were taken for a period of seven consecutive days from 7 a.m. to 9 p.m. Traffic classified as to vehicle, destination, farm-town, interurban, intercounty, interstate. In 1917 traffic counts at 107 stations. See Iowa State Reports for 1917, 1918, 1919, and Bulletin of Engineering Experimental Station.

MAINE.

Traffic counts taken from 1916 to date; at present for 44 stations, for one week, the last week in August, and the first week in September.

MARYLAND.

Systematic traffic count at various points on state road system since 1912. At that time there were 50 traffic stations; since then number increased until in 1920 there were 191 stations covering about 1600 miles of highways. Counts taken once every month for one-day periods.

MASSACHUSETTS.

Four traffic counts, in 1909 (238 stations), 1912 (156 stations), 1915 (192 stations), 1918 (57 stations). Counts for seven consecutive days in August and again in October. See report Massachusetts Highway Commission, 1918, pages 50–57, also appendix J.

NEVADA.

Traffic counts, one week each, during latter part of October and first part of November in 1917 and 1920.

NEW HAMPSHIRE.

Traffic count during 1918 at 57 stations for seven consecutive days during June, July, August and September. See State Highway Report, 1917–18.

NEW YORK.

Some traffic counts on state highways in 1909, 1914 and 1916. A four-day census from 8 a.m. to 8 p.m. about September 1, on practically all of important highways of state.

RHODE ISLAND.

Traffic counts in 1907, 1908, 1913, 1915 and 1920, at 19 stations on seven consecutive days from August 1, during 1913, 1915 and 1920.

WISCONSIN.

Traffic counts at 100 stations for 1918, 1919, 1920.

13.122 Maintenance

INDIANA. *Purdue University.* Comparative cost studies of maintenance on section of the Jackson Highway west of Lafayette, Ind.

Projected.

OHIO. *Huron County.* Effect on maintenance costs of reinforcement in concrete pavements.

Data of surface maintenance costs in 1920 of brick and concrete pavements laid in the period 1912 to 1919.

WISCONSIN. *Milwaukee County.* Effect of various reinforcements on cracking of surfaces.

Experimental road built in 1917. Reinforcement rib metal, from 3 to 6 lbs. per foot of road. Slabs not reinforced 6" thick at sides and 8" thick at center. Reinforced sections vary from 5" to 7" thick at sides and 7" to 8" thick at center. Last inspection made July 7 and 8, 1920. Effect of reinforcement on cost of maintenance. Results of inspection made in 1917 of 100 miles of concrete road. See report of H. J. KUELLING.

13.222 Operating Expenses

CONNECTICUT. *State Highway Commission.* Operating costs of motor vehicles over types of roads.

To determine effect of type of road on operating cost. Vehicles are operated continuously until they are worn out. Data of operating costs recorded. 1922.

INDIANA. *Purdue University.* Truck operating costs.

To determine actual total costs of operating various types of motor trucks in various lines of business over different types of roads. Copies of cost records are secured from operators who are keeping accurate records. Recording blanks supplied to operators not keeping records who are willing to secure data. Cooperating agency, National Commercial Haulers' Association, Tom Snyder, Secretary. May, 1922.

13.4 Economic Life of Road

PENNSYLVANIA. *State Highway Department.* Condition surveys; economics, width and superelevation.

To determine adequate standards of design. Condition surveys of roads constructed in accordance with standard design of Pennsylvania State Highway Department.

14.4 Curvature

NORTH CAROLINA. *State College of Agriculture.* Highway spirals.

University of North Carolina, New and Simple Method for Laying Out Transition Spirals for Sharp Curves on Highways. Complete tables ready for publication.

VIRGINIA. *Polytechnic Institute.* Transition curves.

New curves and special tables. To simplify the work of laying out transition curves.

14.6 Tractive Resistance

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Tractive resistance of dirt roads.

See U. S. Department of Agriculture, Office of Road Inquiry, Bulletin No. 20.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Torsion transmission dynamometer for tests of tractive resistance.

U. S. Department of Agriculture Circular No. 72.

DISTRICT OF COLUMBIA. *U. S. Bureau of Standards.* Operating characteristics of vehicles.

Investigation undertaken for purpose of showing advantage to be gained in reduction of fuel consumption by changes in the intake system of automobiles.

CONNECTICUT. *Yale University, Sheffield Scientific School.* Tractive resistance of motor vehicles.

Measurement of tractive resistance of vehicles by laboratory apparatus, principally for comparison of tire resistance under different conditions. October, 1916, to April, 1922. Rear wheel dynamometer 5.6 ft. diameter, with instrument for measuring speed, drawbar pull, etc. A combination of transmission and absorption dynamometers attached to drum shaft, whose joint use permits of sensitive measurement of rolling resistance of the vehicle. Copies of working plans are available in typewritten form with blue-prints and photographs. Published in part in *Automotive Industries*, N. Y., 1922. To be published in full in Society of Automotive Engineers Journal, at later date.

ILLINOIS. *Portland Cement Association, Chicago.* Tractive resistance of roads to motor trucks.

Tests on earth, gravel, bituminous macadam, brick and concrete road sections, using five 2-ton trucks. Tests made with trucks both empty and loaded. See, Gasoline Consumption Tests Demonstrate Value of Hard Smooth Surfaced Roads, by A. N. Johnson, Engineering News-Record, Nov. 7, 1918.

IOWA. *State College.* Investigations of the resistance to translation of motor vehicles. 1920–22.

Accelerometer, space-time curve recorder, gasoline flow-meter, trucks, touring cars, etc. Cooperating agencies, Iowa State Highway Commission, U. S. Bureau of Public Roads.

KANSAS. *State Agricultural College.* Atmospheric resistance to translation of motor vehicles.

The determination of the proportion of the total resistance to be overcome by motor vehicles that is due to atmosphere. The principal apparatus is a float upon which vehicles are exposed to the wind. The pressure of the wind against the vehicle is measured and recorded while simultaneous wind velocity readings are taken. Date of beginning: spring of 1921. Cooperating agency: U. S. Bureau of Public Roads.

MASSACHUSETTS. *Quartermaster Corps, U. S. Army.* Tractive resistance of roads.

To analyze gross tractive resistance into its elements; to devise suitable measuring instruments for the work; to determine the rolling resistance of several classes of road surfaces to various motor vehicles, as affected by load, speed, tire equipment, temperature, etc. Passenger automobiles, motor trucks, and trailers are tested. Internal resistances of vehicles are determined on a drum dynamometer in Mason Laboratory of Yale University. Gross tractive resistance is found by road tests, the coasting decelerations being measured by a Wimperis or Massachusetts U-tube accelerometer. For sliding friction and low speed tractive resistance tests, a Gulley Tractor Dynamometer is used. Tests were begun September, 1921. Cooperating agencies: National Research Council, U. S. Bureau of Public Roads, Massachusetts Department of Public Works, Connecticut Highway Commission, Harvard University, Massachusetts Institute of Technology, Yale University, Society of Automotive Engineers. See Highways Green Book, 1922.

MICHIGAN. *University of Michigan.* Tractive resistance of roads.

Four Packard trucks, with rear transmission system removed, towed on concrete and gravel roads. (See Economic Grades.) Cooperating agency: Michigan State Highway Commission.

14.7 Economic Grades

IOWA. *State College.* Determination of economic grades for highways.

Special apparatus for recording fuel consumption and speed of vehicle. Flow meter, accelerometer, etc. Vehicles are operated over grades under various conditions of braking, clutching, and declutching. 1919–22. Cooperating agency, Iowa State Highway Commission.

MICHIGAN. *University of Michigan.* Investigations covering economic grades.

To determine relation between capital cost and operating expenses of highway grades. See 1922 Proceedings of Eighth Annual Conference on Highway Engineering, University of Michigan. On file at Davis Library of Highway Engineering and Highway Transport, University of Michigan.

15.3 Sources of Revenue

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Sources of revenue for highway improvements.

Study of relative importance of sources of highway revenue. Real and personal property, bond issues, motor vehicle licenses, special assessments (Road districts), gasoline taxes, general sources—income taxation, federal aid, legislative appropriation.

15.311 Types and Costs

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Effect of highway improvements upon land values.

Development of method of determining influence of highway improvement on rural land values. Analysis made in four Wisconsin counties.

NEBRASKA. *University of Nebraska.* Economic value of Nebraska highways.

Changes in value of farm land as a function of distance from nearest state highway. Between years 1915–1921, inclusive.

16. BIBLIOGRAPHY

American Society of Civil Engineers. Bibliography of highway engineering.

A subject catalogue of the books and pamphlets in the Engineering Societies Library, December, 1921. See Proceedings of American Society of Civil Engineers, Vol. 48, No. 1, January, 1922.

Highway Education Board, Willard Building, Washington, D. C. Bibliography on highway transport.

Quartermaster Corps, U. S. Army. Digest of literature on tractive resistance, by Quartermaster Tractive Resistance of Roads Research, edited by Norman B. Van Patten, Asst. Librarian, Massachusetts Institute of Technology. See **14.6.**

20. OPERATION

21. CONTROL OF TRAFFIC

21.1 Handling Traffic

CONNECTICUT. *Eno Foundation for Highway Traffic Regulation.* Study of various systems of handling highway traffic.

At roadway foci, junctions and intersections, by means of rotary system, isles of safety.

21.4 Police Regulation

CONNECTICUT. *Eno Foundation for Highway Traffic Regulation.* Studies of the effect of highway traffic regulation in controlling vehicular and pedestrian traffic on the highways efficiently and safely.

The Eno Foundation is incorporated under the laws of Connecticut by William Phelps Eno, of Washington, D. C.

ILLINOIS. *Chicago Department of Highways and Streets.* Ordinance governing size and weight of motor equipment.

Protection to highways. Contemplated design of stronger pavement. Change of old ordinance to meet modern equipment requirements. Date of beginning, 1920. Cooperating agencies: Automobile Trades Association, Chicago City Club.

21.5 Traffic Guides

CONNECTICUT. *Eno Foundation for Higway Traffic Regulation.* Studies of traffic lines, signs, standards, bumpers, turning caps, semaphores, crowsnests, lights, etc., which can be economically and should be extensively employed to guide the activities of both pedestrians and drivers.

24. MAINTENANCE

24.2 Methods

24.21 Joint and Crack Filling

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Investigation of joint fillers.

(a) To determine desirable properties of prepared joint fillers. Analysis of present commercial joint fillers; (b) To develop a filler having same color as concrete. Laboratory and field experiments with possible materials. Date of beginning, June 1, 1922.

NEW YORK. *Cornell University.* Elastic properties of joint filler for concrete roads.

Date of beginning, January, 1922. Cooperating agency, New York State Highway Commission, Hornell, N. Y.

24.22 Surface Treatments

WISCONSIN. *State Highway Commission.* Maintenance of gravel roads.

To study maintenance of gravel roads, not surface treated, having a fairly light surface (3 to 6 in.) but built wide with fairly flat crown (4" in 30'). An experimental road has been built 2½ miles long, having surface of 3" to 6" of gravel 30 ft. wide, 4" crown. In 1922 one mile treated with Tarvia B, special treatment; 0.85 mile with calcium chloride; remainder as originally built. Traffic on this road during summer months is 1500 vehicles per day.

24.2211

ILLINOIS. *Department of Public Works and Buildings.* Investigation of road oils for use in various sections of Illinois.

To determine suitable road oils for use in various types of soil. Field and laboratory observation. The dust-laying and waterproofing effect of road oil on the different soil types of Sangamon County; the effect of oil on the capillary movement of subsoil water under various weather and drainage conditions. Date of beginning, February 10, 1922. Cooperating agency, Superintendent of Highways, Sangamon County, Illinois.

ILLINOIS. *University of Illinois.* Field tests of road oiling.

Determination of (1) relative merits of different kinds of oils, (2) best method of applying oil, amount and number of applications per year, (3) best method of maintaining an oiled road, (4) cost of well maintained oil road per year, (5) number of days of good surface per year obtained by properly oiling and maintaining an earth road. Date of beginning, October, 1921. Cooperating agencies: road machinery companies, oil companies, and Champaign County.

MAINE. *State Highway Commission.* Bituminous surfaces on gravel roads.

Gravel roads treated with refined tar product suitable for cold application in comparison with various asphaltic oils and tar products. Effect of freezing and thawing. Methods of application. Costs. Traffic limitations.

OKLAHOMA. *Muskogee. Department of Highways and Streets.* Wearing qualities of carpet treatment with asphaltic oils.

Liquid asphalt, etc., on clay-gravel roads and topped with mine chats.

24.2212 Dust Preventives

MICHIGAN. *University of Michigan.* Investigations covering dust prevention and preservation of gravel roads.

See 1922 Proceedings of Eighth Annual Conference on Highway Engineering, University of Michigan. On file at Davis Library of Highway Engineering and Highway Transport, University of Michigan.

MICHIGAN. *State Highway Department.* Efficiency of various dust preventives.

To determine the relative merits of oils, tars, and calcium chloride as dust palliatives for gravel roads. Field tests ½-mile sections. Completed 1921.

PENNSYLVANIA. *State College.* Dust prevention by the use of palliatives. See Bulletin No. 23, 1915.

24.24 Reshaping

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* A study of gravel roads.

Investigation to determine behavior of different types of gravel roads under traffic conditions; to study the phenomena of rhythmic corrugations and to compare various methods of maintaining gravel roads with special reference to the elimination of such corrugations.

IDAHO. *Department of Public Works.* Design of maintenance equipment: sand distributor, road plane, patrol maintainer, highway siphon.

VERMONT. *Highway Department.* Cause of washboard or corrugated gravel road.

To learn the cause of corrugations. Factors influencing their formation and ways and means of controlling or eliminating their formation. Ordinary laboratory equipment. Comparative samples and tests to determine effect of moisture, effect of clay. Collection of data as to roads affected and those not affected. Date of beginning, July, 1921.

24.3 Snow Removal

WYOMING. *State Highway Department.* Elevation of grade line to eliminate snow blockade.

30. DESIGN (ROAD)

31. SUB-SOIL STUDIES

31.1 Properties

CALIFORNIA. *University of California.* Adobe sub-soils.

Tests of adobe sub-soils collected from various localities along the highways of the state of California. Also a study of the adobe treatment on the test highway at Pittsburg, Calif. The laboratory of the Department of Civil Engineering has in its possession numerous samples from various parts of the state left from studies of 1920–21 and at an early date intends to continue its adobe inquiries. First results printed in report on California State Highways by the Automobile Club of Southern California, and the California State Automobile Association, January, 1921. Begun September, 1920. Completed February, 1922.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Physical properties of subgrade materials.

To determine the physical and chemical characteristics of subgrade materials as correlated with their behavior under service conditions. Samples of the soil are obtained from portions of the subgrade where subgrade failure has taken place and from adjoining portions of the subgrade on the same road which has not failed. Samples should be taken and supporting information obtained as follows:

Samples of one cubic foot each are taken from the subgrade, preferably directly under the failure, photographs should be taken of the site of failure, and complete information given regarding the topography, preferably in the form of cross-sections and profile of the road in the vicinity of the failure. Notes are taken regarding the character of the failure, the approximate amount of traffic, drainage conditions, the probable presence of water-bearing strata, or other conditions that may make for the wet condition of the subgrade. Data are given tending to show how water reached the subgrade, whether by vertical capillarity, horizontal capillarity, through seepage strata, from the surface, etc. An auger 1½ inches in diameter is useful to explore the underlying subgrade and much useful information may be obtained by noting the texture and condition of moisture of the soil at different depths.

Similar samples and observations are taken from spots in the same road which have not failed. If the texture changes down to a depth of 5 or 6 feet, as determined with the auger, a one-quart sample is taken and placed in small cloth bag.

Laboratory procedure is carried out in accordance with methods described in the paper entitled "Tests for Soils with Relation to Their Use in the Subgrade of Highways," by A. T. Goldbeck and F. H. Jackson, Proceedings A. S. T. M., 1921.

In addition to the samples collected as described, artificial subgrade materials are made up containing a wide range in variation in content of colloidal material, silt and sand and the complete laboratory tests in order to establish, if possible, the enter-relation of these tests with materials having wide variation in characteristics.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Special studies of vertical and horizontal capillarity of materials having definite physical and chemical characteristics.

To obtain and correlate data on horizontal and vertical capillarity of soils. The method of procedure followed is that described in Department of Agriculture Bulletin No. 835 entitled *Capillary Movement of Soil Moisture.*

GEORGIA. *School of Technology.* Capillary action in Georgia soils.

Capillary tubes, mechanical analysis screens, bearing power spiral machine. The flow of water by capillary action per day was obtained for 30 days in different Georgia soils. The bearing power with different percentages of water was also obtained by finding load required to sink a 2″ cylinder ¼″ into the soil. The mechanical analysis of soil was also taken. July 1920–July 1921. Cooperating agency: Georgia State Highway Department.

INDIANA. *Purdue University.* Studies of sub-soil drainage.

To determine the effect of moisture and temperature on soils and to study the capillary action of soils. Various soils, glass tubing, machines for molding, cementation test cylinders and measuring instruments. Various kinds of soils ground to dust, to which different percentages of water have been added, are molded into small cylinders. These cylinders are subjected to different temperatures, including freezing, and the results noted. Then they are dried to determine the change in size due to moisture content. The dry cylinders are next placed in water to note the speed and effect of absorption. It is desired to determine temperature and moisture coefficients of expansion, and to outline a method of procedure for more extended tests. Date of beginning, February, 1922.

IOWA. *State College.* Soil water, and soil water movements as related to drainage.

A study of the movement of water through various soils with special reference to effect on farm crops and road drainage. Date of beginning, 1920.

NORTH CAROLINA. *University of North Carolina.* Sub-soil investigations.

MEMORANDUM OF SUB-GRADE COMMITTEE OF FEDERAL HIGHWAY COUNCIL

Chairman, Senator Coleman du Pont; Vice-Chairman, Mr. C. M. Upham

This sub-grade committee is divided into ten problem committees, about as follows:

Committee No. 1. Collection of samples; tests for physical properties. Professor F. H. Eno, Chairman.

Committee No. 2. Machines for testing the bearing power.

Committee No. 3. Studies of drainage. Mr. Frank Rogers, Chairman.

Committee No. 4. Chemical treatment of sub-grades. Prof. C. M. Strahan, Chairman.

Committee No. 5. Mechanical treatment of sub-grades. Mr. Paul E. Sargent, Chairman.

Committee No. 6. Freezing and thawing of soils. Professor H. J. Hughes, Chairman.

Committee No. 7. Volume changes in soils. Professor F. E. Giesecke, Chairman.

Committee No. 8. Pressures on sub-grade. Mr. Robert A. Cummings, Chairman.

Committee No. 9. Classification of sub-grade materials. Mr. Joseph Hyde Pratt, Chairman.

Committee No. 10. Test of models and full sized sections. Professor F. H. Eno, Chairman.

31.2 Bearing Power and Volume Change

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Bearing value tests of soils in field.

Development of suitable field bearing value test. Comparison of bearing value as found in field and as ascertained by laboratory methods. Date of beginning, 1921.

NEW YORK. *American Society of Civil Engineers.* Special committee on bearing value of soils.

Questionnaire August, 1921. Progress report, Proceedings, March, 1922, p. 523.

OHIO. *State University.* Soil survey relative to road failures.

To obtain sufficient knowledge of soils so that the proper treatment may be given to road subgrades, in order that they may uniformly support the roads built upon them. Samples of soils collected at location of pavement failures. 1. For mechanical analysis the sieve and centrifuge method. 2. For bearing value, use 4½ dia. heads, wt. 10 lbs., with drop weight of 10 lbs. Shape of heads: (a) flat cylinder, (b) cone, 30° with horizontal, (c) sphere, all to be checked against A. S. T. M. standard static test. 3. Saturation limit of soil, method not yet determined. See Bulletin No. 61, Ohio Good Roads Federation, First Annual Report on Survey of Road Failures, by F. H. Eno, Director of Research, January 19, 1921.

31.21 Moisture Content

ILLINOIS. *Department of Public Works and Buildings, Division of Highways.* Moisture content and bearing value of subgrade under Bates road.

Tested at various seasons.

SOUTH DAKOTA. *University of South Dakota.* A comparative study of the moisture content of soils in tiled and untiled subgrades below hard surface paving.

SOUTH DAKOTA. *University of South Dakota.* Minimum amount of moisture in subgrade that will cause heaving due to frost.

TEXAS. *University of Texas.* Effect of moisture on volume changes in soils.

To determine volume changes of soils occurring with changes of moisture content. One level bar and five rods having plates attached to one end for each station. At each station to be observed, metal base plates with rods attached will be placed at depths from two to ten feet below the surface. Levels will be taken on the upper ends of each of these rods by a level bar reading to one-thousandths of an inch, while at the same time samples of soil will be taken from the same depths as the plates for determination of moisture content. A relation between the moisture content and movement of the rod will be sought. January 1, 1922, to September 30, 1923. Cooperating agencies: U. S. Bureau of Public Roads and Texas State Highway Department.

31.22 Special Treatment

CALIFORNIA. *Highway Commission.* Experimental treatment of adobe subgrade on section of state highway, being constructed at Solano County between Denverton and Rio Vista.

In order to test different methods of treating adverse sub-soils in order to improve their support for paving, the California Highway Commission will institute a series of experiments on a section of road now being constructed from Denverton to Rio Vista in Solano County. After the contractor has finished the rough grading, including the shaping but not the rolling, the grade to a width of 21 feet when thoroughly dry will be ploughed with a rooter or sub-soil plough until the soil is loosened to approximately the depth to be treated, after which it will be thoroughly harrowed, disked or rolled until the soil is well pulverized. These nine 500-feet sections will then be treated as follows:

Section 1. Mixed with Portland Cement to depth of 12", ratio 1:10.
Section 2. Mixed with Portland Cement to depth of 12", ratio 1:20.
Section 3. Mixed with Portland Cement to depth of 6", ratio 1:10.
Section 4. Mixed with Portland Cement to depth of 6", ratio 1:20.
Section 5. Mixed with hydrated lime to depth of 12", ratio 1:20.
Section 6. Mixed with pulverized limestone to depth of 12", ratio 1:20.
Section 7. Planed and harrowed to depth of 12".
Section 8. Planed and harrowed to depth of 12" and 60% asphaltic road oil spread at rate of 3-5 gallons per sq. yd.
Section 9. Planed and harrowed to depth of 6" and oil applied as in section 8.
Cooperating agency, U. S. Bureau of Public Roads.

District of Columbia. *U. S. Bureau of Public Roads.* Studies of the chemical treatment of soils.

To investigate the possibility of so changing the characteristics of certain types of soils by admixtures that their bearing value will be materially increased or their drainage will be rendered more effective. Soils of known high plasticity mixed with various percentages of the following materials:

1. Portland Cement.
2. Ground granulated blast furnace slag.
3. Hydrated lime.
4. Mixture of ground granulated blast furnace slag and hydrated lime.
5. Ground slag and "Cal" (calcium oxychloride).
6. Asphaltic materials.
7. Alum.

The following tests are then made: 1. Moisture equivalent. 2. Capillarity. 3. Shrinkage. 4. Retentivity. 5. Bearing value. 6. Absorption. 7. Field studies. Date of beginning, 1920.

Illinois. *Lewis Institute, Structural Materials Research Laboratory.* Earth concrete; use of lean mixtures of cement and earth for subgrade improvement.

Illinois. *University of Illinois.* Laboratory tests to determine the effect of physical properties of soil upon oil absorption.

To determine the effect of the physical condition of soils upon their ability to absorb oil; to determine the relation between the quantity of oil applied and the waterproofing effect accomplished; to determine the relation between the method of applying the oil and the waterproofing effect accomplished; and to determine results other than waterproofing which may affect value of road oil. Date of beginning, March 1, 1922.

OREGON. *State Highway Commission.* Sub-soil investigations.

To determine the conditions of subgrade under pavements of various types and to observe the effect of drainage, soil treatment, shoulder construction, etc., upon different classes of soil. Samples obtained with soil augers. The laboratory work is similar to that outlined by Mr. A. T. Goldbeck. Seventy sampling stations were established on about 300 miles of paved highway, covering many different soil classifications and types of pavement. When started, two samples from each station were taken every ten days, one being from the shoulder and one from well under the pavement. However, the work has been suspended indefinitely, after running about two months. Date of beginning, February, 1922.

SOUTH DAKOTA. *University of South Dakota.* Effect of subgrade consolidation on the transmission of heat and moisture.

31.24 Climatic Changes

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* The effect of temperature change on capillarity.

Artifically graded soils having widely varying characteristics placed in tubes 2½ inches in diameter by 36 inches long, and subjected to freezing action after which moisture content of soil in different parts of tube is determined. Date of beginning, 1920.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Study of the effect of freezing and thawing on volume changes and supporting properties of soils.

Artifically prepared samples of clays, silts and sand are tested for bearing power in static bearing power testing machine both before and after being subjected to freezing action. Volume changes are noted both after freezing and thawing. Time of beginning, 1920.

31.3 Distribution of Pressure

NEW YORK. *Rensselaer Polytechnic Institute.* Thrust in a granular mass.

To study the lateral and vertical thrust of a mass of sand on a vertical wall, its amount, direction of its line of application, and the location of the resultant thrust. To study the effect of the size of grains of the material under investigation and the effect of a vertical load in the vicinity of the wall. To measure internal angle of friction. Goldbeck soil pressure cells, a wooden tank 6′ x 4′ x 5′, indicating apparatus, air compressor, and accessories. Load application is accomplished by the usual method of a bottomless box, but an additional device is used to determine the effect of wall friction of the box. Horizontal thrust measurements were obtained by a pressure indicating apparatus similar to that used by Goldbeck, but gauge readings were in inches of water or mercury. Date of beginning, November, 1921. Cooperating agency: Manning Abrasive Co.

NEW YORK. *American Society of Civil Engineers.* Transmission of pressures in ballast.

Progress report of the Special Committee to Report on Stresses in Railroad Track Transmission. A. S. C. E. 1919–20.

OHIO. *University of Cincinnati.* Earth pressures; design of apparatus.

See Engineering News-Record, August 1921; January, 1922.

OHIO. *University of Cincinnati.* Earth pressures; theory and tests.

Thesis for Ph. D., June, 1922.

PENNSYLVANIA. *State College.* Experiments on distribution of vertical pressure in earth.

See Bulletin No. 8, 1913; Bulletin No. 11, 1914.

32. DRAINAGE

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Studies of moisture content under road surfaces with different types of drainage.

To determine the efficiency of different types of drainage for different conditions of subsoils and of topography. Different types of drainage are to be installed in experimental roads and moisture samples of the subgrade taken at different seasons in order to determine which of the various drainage schemes tried is most efficient. In the construction of the pavement special openings can be prepared through which samples of the subgrade may be obtained. Among the studies to be made are:

1. Size and location of side-tile drains, both as regards depth of drains and position with respect to center line of pavement. 2. Spacing and depth of herring-bone drains. 3. Studies in water-proofing of shoulders. 4. Studies in water-proofing of subgrade. 5. Studies in grading up of subgrade. Date of beginning, 1920.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* The influence of topography on hydraulic gradient of ground water.

A study to throw light on the proximity of ground water to the surface as influenced by the topography. A field experiment in which observations are made on actual roads laid under different topographical conditions. Soil samples are obtained at different depths by means of a soil auger at various distances at each side of the road surface, so that profile lines of equal moisture content may be drawn.

IOWA. *State College.* Erosion of open ditches.

To determine the factors causing erosion in open ditches and methods for its prevention. Current meters, surveying instruments, etc. Date of beginning, 1920.

IOWA. *State College.* Road drainage.

Cooperating agency: Iowa Highway Commission. Determination of hydraulic gradient across the right of way by means of vertical pipe set in the soil.

IOWA. *State College.* Runoff from large drainage districts.

To gather data on the runoff from large drainage districts, both tile and open ditch. Weirs, recording gages, etc. Date of beginning, 1916.

MISSISSIPPI. *Geological Survey.* Drainage investigations in part affecting conditions of road building, especially in the Mississippi Delta.

OKLAHOMA. *Muskogee, Oklahoma Department of Highways and Streets.* Sub-tile drainage of city paved streets.

PENNSYLVANIA. *Pennsylvania State Highway Commission.* Designs of under-drainage.

To determine proper arrangement of under-drains, by observing results obtained where various Pennsylvania Highway Department standard types of drainage structures have been employed.

32.3 Culverts

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Studies in the treatment of concrete drain tile to prevent alkali action.

A section of drain tile is immersed for 20 minutes in crude water-gas tar, removed from bath and allowed to dry for three days. The sample is then weighed, placed in a 3 per cent solution of magnesia sulphate and sodium sulphate and allowed to remain for two weeks. It is then removed and allowed to dry for two days and examined microscopically to detect the effect, if any, of alkalies on the tile. It is then re-weighed and the loss, if any, noted. This procedure is repeated at monthly intervals for six months, when the sample is placed in distilled water for one week and then dried, weighed, and the percentage loss in weight noted. A check sample of untreated tile is tested in the same way and the difference in weight compared at the end of the six-month period. Date of beginning, May, 1921

DISTRICT OF COLUMBIA. *U. S. Bureau of Standards.* Durability of cement drain tile and concrete in alkali soils.

IOWA. *State College.* Investigation of pipe culverts.

To study the load-carrying capacities of pre-cast pipe culverts. Special loading devices. See Bulletin No. 57, Ames Engineering Experiment Station. Date of beginning, 1915. Cooperating agencies: Iowa Highway Commission; culvert pipe manufacturers.

IOWA. *State College.* Load distribution on culvert pipe in ditches and embankments.

To determine the actual loads upon culvert pipe under varying conditions as to height of tile and nature of materials. Apparatus is special. See report of Dean Marston presented at 1922 meeting of American Society for Testing Materials. 1915–25. Cooperating agencies: U. S. Bureau of Public Roads, Iowa Highway Commission. See Bulletin No. 57, Ames Eng. Exp. Station.

IOWA. *State College.* Methods of reinforcing concrete pipe and tile.

A study of methods now in use. Procedure is same as for regular tile testing. See Bulletins 47 and 57. 1915–20. Cooperating agency: Tile manufacturers.

IOWA. *State College.* Design of portable tile-testing machine.

IOWA. *State College.* Effect of alkali on concrete drain tile.

To study the factors affecting the durability of concrete drain tile when laid in alkali soil. Fragments of concrete drain tile from check samples are subjected to various alkali solutions under laboratory conditions. Frequent observations are made to determine the effect, if any. Date of beginning, November, 1920. Cooperating agencies: American Society for Testing Materials, American Concrete Pipe Association, U. S. Bureau of Standards, U. S. Bureau of Public Roads.

IOWA. *State College.* Effect of freezing and thawing on drain tile.

To determine the effect of alternate freezing and thawing on clay and concrete drain tile. Whole tile frozen out-of-doors at night and thawed at room temperature during the day; also artificial freezing and thawing of tile. See Bulletin 49. Date of beginning, 1916. Cooperating agency: American Society for Testing Materials.

IOWA. *University of Iowa.* Flow of water in corrugated iron pipe.

To secure, develop and make available, information as to the hydraulic properties of culvert pipes of different kinds, particularly vitrified sewer pipe and corrugated iron pipe. Date of beginning, August 1, 1922. Cooperating agency: U. S. Bureau of Public Roads.

34. BASE COURSE

34.1 Concrete

34.13 Proportioning

DISTRICT OF COLUMBIA. *U. S. Bureau of Standards.* Inundated sands.

Measurement of sands in water for concrete mix. To eliminate uncertainty due to bulking effects caused by varying percentages of moisture in sands. Method of measuring water and sand in same vessel. Date of beginning, August 15, 1922. In cooperation with White Construction Company, New York City.

35. TOP COURSE

35.1 Concrete

35.15 Type of Slab

CALIFORNIA. *State Highway Commission.* Experimental concrete base construction on section of state highway in Solano County, east of Fairfield, consisting of precast slab and expansion joint construction with and without dowels.

To try out precast slab method of construction and also expansion joint treatment. Date of beginning, July, 1922. Cooperating agency: U. S. Bureau of Public Roads.

ILLINOIS. *Chicago. Department of Highways and Streets.* Types of pavement for heavy traffic.

Construction of pavement to withstand heavy traffic, and vibration of car tracks. Standard contractors' equipment. Date of beginning, 1919.

35.16 Joints

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Investigation of the merits of reinforcements and special joints proposed or now used for distributing loads across joints or cracks in rigid pavements.

To determine the resistance of edges and corners of rigid road slabs to heavy loads as well as to secure information of the load distributing value of proposed and current joints and systems of reinforcements. A series of road slabs will be constructed with different kinds of joints and subject them to heavy repeated static loads and very light impact loads. The test slabs should be of two kinds, those composed of sections representing cracked road surfaces, and those representing surfaces comprised of specially designed blocks. The test specimens will be constructed in 14′ x 14′ slabs divided into four 7′ x 7′ sections by joints. The joints to be treated in the same manner that joints and cracks are maintained on actual roads. The loads, both static and impact, are to be applied by the impact machine. Date of beginning, July 1, 1921.

ILLINOIS. *Department of Public Works and Buildings, Division of Highways.* Replacement of corrugated metal central joint in concrete roads by a special bituminous joint.

Slabs 4 feet by 6 feet with joints are tested to determine strength of joint.

ILLINOIS. *Department of Public Works and Buildings.* Prevention of bond in distributing bars in concrete roads.

July 18, 1921, to October 24, 1921.

WISCONSIN. *Washington County.* Omitting felt at joints in concrete pavement and using dowel rods at joints connecting the 2-days run.

To prevent one slab raising or lowering above or below adjacent slab. 3′ dowel rods are inserted 18″ in days runs and projecting 18″ for next days run, spaced 6″ from each side and 2″ to 10″ thereafter in 18′ pavement. Holes are drilled through end board at above spacing to hold dowel rods. July 15, 1922, to April 1, 1923. Cooperating agency: Wisconsin Highway Commission.

35.17 Resistance to Traffic and Climate

FLORIDA. *University of Florida.* Action of Florida climate on construction materials.

35.171 Surface Wear

DELAWARE. *State Highway Commission.* Wear of concrete roads.

Wear plugs inserted in road, U. S. Bureau of Public Roads method. Cooperating agency: U. S. Bureau of Public Roads.

MARYLAND. *University of Maryland.* Study of cores cut from concrete roads to determine effect due to traffic.

These cores represent roads that have been subjected to traffic for several years. Core drill engine, water tank, etc., mounted on truck; 100,000-lb. testing machine; small equipment for making caps and preparing cores for tests. Two men on truck securing cores from the road which are sent to the laboratory, where they are capped with cement mortar, and after due aging are run through the testing machine. Date of beginning, summer of 1921. Cooperating agencies: State Road Commission of Maryland, U. S. Bureau of Public Roads.

35.173 Cracking

IOWA. *State Highway Commission.* Control of cracking in concrete pavements.

To determine causes of cracking. Date of beginning, October, 1921.

MICHIGAN. *Wayne County.* Tar paper on subgrade under concrete slab.

NEW JERSEY. *State Highway Commission.* Determination of the effect of subgrade on the quality of concrete and the causes for cracks.

To see if sandy or porous soils do not cause more shrinkage cracks in concrete than denser or less absorbent soils.

NEW JERSEY. *State Highway Commission.* Effect of strength of concrete on the production of cracks.

To determine whether the strength of concrete has a tendency to increase or decrease transverse cracks.

NEW JERSEY. *State Highway Commission.* Effects of rapid curing on production or prevention of cracks in concrete pavements.

To see whether a slow or rapid-setting cement will increase or decrease transverse cracks in concrete.

PENNSYLVANIA. *State Highway Department.* Causes of hair cracking in concrete pavements.

To find reason for their existence and a method of preventing them. Newly made sections of concrete submitted to different curing conditions during the first twenty-four hours.

35.1741 Loads, Static

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Effect of impact on road surfaces.

To obtain relation of the resistance of various types of road surfaces to impacts such as might be expected from moving motor truck wheels. Test consists of breaking about 150 slabs 7 feet square representing surfaces now used for highways. Slabs are laid on specially prepared subgrades. For results from tests on first 50 specimens, see *Public Roads*, October and November, 1921.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Loads on pavements produced by motor vehicles.

Trucks of various capacities with various tire equipment driven at several speeds were allowed to pass over obstacles and produce a blow upon an apparatus which registered the severity of the blow by means of the deformation of a copper cylinder. The latter was afterward calibrated in an impact testing machine to compute the foot-pounds of energy delivered by the vehicle to the road. This information was afterwards correlated with impact tests of concrete slabs. Date of beginning, 1919.

INDIANA. *Purdue University.* Resistance of concrete to tire loads.

To determine the resistance of concrete surfaces after they have been subjected to tire loads. The surface resistance of mortar slabs is measured by the load required to push therein a one-half inch steel ball to one-half of its diameter before and after the application of a solid rubber tire loaded with increasing loads. Cooperating agency: U. S. Bureau of Public Roads.

MICHIGAN. *City of Detroit.* Deflection, pressure, distribution and contact areas in cushion rubber tires, 36″ x 5″.

Loads up to 7,000 lbs. on dual tires.

NORTH CAROLINA. *State Highway Department.* Impact and static load test of concrete slabs.

To compare different types of pavements. Special impact tests. Comparison of impact and static load test by means of laboratory-made slabs.

35.1742 Loads, Impact

CONNECTICUT. *Engineering Mechanics Department, Yale University.* Inertia forces in vehicles.

To determine the increase in vertical reactions (akin to impact force) in vehicle wheels resulting from starting and stopping. Specially built rubber-tired model for laboratory demonstration; scales, etc. Also (later) motor cars and trucks. Date of beginning, May, 1922.

ILLINOIS. *Department of Public Works and Buildings, Division of Highways.* Relation of impact load to static load on concrete pavements.

Deflections of a pavement slab under the impact of a truck with a known wheel load moving over various obstructions on the road surface are measured. The slab is then loaded statically at the point of impact application until the same deflection of the slabs is obtained.

35.175 Contraction and Expansion

DISTRICT OF COLUMBIA. *U. S. Bureau of Roads.* Causes and movements producing stress in concrete road surfaces during time of setting and thereafter.

To secure information on the movements which occur in rigid surfaces immediately after construction and thereafter. The apparatus necessary for making the above investigation includes Ames dials, strain gages, and devices for autographically recording desired observations of movements and temperatures and moisture conditions. Extended observations made on special slabs of different sizes and thicknesses constructed at Arlington with devices installed to allow continuous readings throughout the year. Date of beginning, July, 1921.

KANSAS. *State Agricultural College.* Temperature stresses in rigid pavement slabs.

Plugs set in new pavement; 20-inch Berry Strain gage; differential bar. Measurement of change of distance between plugs on center line of pavement and at right angles thereto, and determining stress set up by expansion of slab. The differential bar was used to compute the true zero length between gauge points. Date of beginning, August, 1920. Cooperating agency, Kansas Highway Commission.

35.176 Warping

ILLINOIS. *University of Illinois.* Warping of concrete road slabs.

Determination of magnitude, cause, and effect of warping of road slab. Thermocouples for measuring temperatures, subgrade cylinders for measuring vertical movements of slab at different points. Goldbeck pressure cells for measuring distribution of earth pressure on bottom of slab. Date of beginning, December, 1921.

35.18 Distribution of Load to Subgrade

CALIFORNIA. *University of California.* Pressure cells.

To study the behavior of subgrade under traffic. Mechanical Department of the Laboratory has constructed thirteen pressure cells of the Goldbeck type. These were installed April, 1922, under special slabs built into the Pittsburg Test Highway at Pittsburg, California. The cells have been installed next to deflection apparatus. Date of beginning, April, 1922.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* A study of the distribution of pressure to the subgrade through different thicknesses and different types of surfaces.

To determine the maximum intensity and distribution of pressure on the subgrade in order to calculate the bending moment in rigid slabs and the thickness of the non-rigid type required to distribute the load so that the intensity of pressure produced will be less than the bearing value of the soil. Upon a subgrade having a known definite water content are placed different thicknesses of concrete slabs and different thicknesses of stone bases. Upon these slabs, which are ten feet square, are distributed special apparatus for measuring soil pressures. Loads are applied to the slabs in the most disadvantageous position (a position which will produce the maximum intensity of pressure), and the distribution of pressure is noted when known loads are applied to the surface of the slabs. Date of beginning, 1921.

35.194 Reinforcing

DISTRICT OF COLUMBIA. *U. S. Bureau of Standards.* To determine relative values of expanded metal and of loose bars as reinforcement for slabs supported on two sides.

Stresses measured with strain gage. From 1918 to 1919. Consolidated Expanded Metal Companies furnished metal. Lehigh University cooperating. Report ready for press, should be out about January, 1923.

MINNESOTA. *University of Minnesota.* Structural behavior of road slabs with reinforcement.

NEW YORK. *Schenectady Department of Highways and Streets.* Changing design of reinforced concrete pavement.

NORTH CAROLINA. *State College of Agriculture and Engineering.* Comparison of plain and reinforced concrete slab pavements.

PENNSYLVANIA. *University of Pennsylvania.* Investigation of reinforced concrete highway slabs 18 feet square.

a. Effect of variable amount of reinforcing steel. b. Study of earth pressure under concentrated loads. To find out if there is need of greater percentage of steel in road slabs when they resist concentrated loading. To study area of slab taking positive loading. Heavy cross beam tied to 28,000 lb. steel rails, loading jacks, deflection apparatus, twenty earth pressure cells, strain gages. Slabs built to standard Pennsylvania State Highway Department specifications, except for varying steel. After six months loading was attempted. Temperature affected results. Negative pressures caused some cells to fail so all are being drilled out of road. Will be reset with initial pressure to give a chance to observe negative pressures. November 25, 1921–July, 1922. Cooperating agencies: Pennsylvania Railroad, Pennsylvania State Highway Department, University of Pennsylvania, etc.

WASHINGTON. *City of Seattle, Engineering Department Laboratory.* Transverse test on monolithic brick and concrete pavement slabs.

To determine the difference in transverse strength between concrete slab and monolithic brick of same thickness. Slabs for transverse test 17″ x 48″ with varying thickness; 6″ x 12″ cylinders. March 21, 1918–April 25, 1918.

35.24 Filler

IOWA. *State College.* Investigation of bituminous fillers for brick pavements.

To coordinate results of chemical tests with physical properties of fillers suitable for Iowa. Special apparatus. Date of beginning, 1920. Cooperating agency: The Texas Company.

MICHIGAN. *University of Michigan.* Efficiency of lugless brick used with asphalt fillers.

See 1922 Proceedings of Eighth Annual Conference on Highway Engineering, University of Michigan. On file at Davis Library of Highway Engineering and Highway Transport, University of Michigan.

35.3 Bituminous Concrete, Bituminous Macadam, and Sheet Asphalt

NEW JERSEY. *Atlantic County.* Method of binding bituminous top to concrete base.

To utilize present concrete roads for foundation purposes, later on. To ascertain the requirements for a mechanical bond, in terms of the roughness of the concrete surface and absence of lubrication, etc., thereon. Date of beginning, Jan. 1, 1922.

35.35 Resistance to Traffic and Climate

TEXAS. *Agricultural and Mechanical College.* Bituminous pavement investigation.

To determine why some pavements are successful and others not. Collection of samples of existing pavements and their subsequent analysis in the laboratory, followed by a study of results. See Bulletin 24, Texas Engineering Experiment Station. 1918–1921. Cooperating agency: City engineers of various cities of Texas.

35.355 Shoving and Waving

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Study of properties of bituminous surfaces on concrete and macadam rendering them most stable under traffic.

To study the effect of traffic upon bituminous concrete and macadam by means of an accelerated field test, endeavoring to secure data on desirable characteristics which materials and pavements should possess to resist displacement and wear. Apparatus has been constructed to provide motor truck traffic upon a circular track 560 feet in circumference, comprising a number of sections of pavement. Automatically controlled traffic will be operated over the pavement and observations made of the behavior, such as amount of wear and change in contour. During construction, tests made on the materials and on finished pavements where tests can be applied. Subsequent tests to detect changes in size and grading of aggregate, density, etc., made. Sixteen different sections having for bases concrete, bituminous concrete, and water-bound macadam, with different kinds of bituminous tops, to be observed. Date of beginning, 1921. Cooperating agencies: The Asphalt Association and American Association of State Highway Officials.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* The cause of shoving of sheet asphalt and asphaltic concrete pavements.

To ascertain the causes of the displacement of asphalt surfaces under traffic. Correlation of all available information of any possible value in the study concerning selected pavements which have shoved, wholly or in part, together with a number which have successfully withstood traffic. Duplicate samples taken from significant points in the pavements of Washington, Baltimore, Detroit, Philadelphia and New York. Specimens which were secured by use of a Calyx core drill were examined in both the laboratories of the Bureau and of the cities interested. Additional information was secured from records of the original construction. Date of beginning, July, 1921. Cooperating agencies: The Asphalt Association, Cities of New York, Philadelphia, Baltimore, Detroit, and the District of Columbia.

NEW JERSEY. *Atlantic County.* Method of distributing bituminous paving materials over a cement concrete base to minimize waviness.

35.4 Water Bound Macadam and Gravel

35.41 Materials

MICHIGAN. *University of Michigan.* Efficient binding materials for different kinds of gravel.

OKLAHOMA. *State Highway Department.* Mixture of sand, clay, and road oils as a possible binder for highway surfacing material.

35.5 Top-soil and Sand-clay

GEORGIA. *National Lime Association.* Use of lime in sand-clay roads.

GEORGIA. *State Highway Department of Georgia.* Field construction experiments in connection with current construction by the state and counties of Georgia.

To try out special mixtures of road soils and to test the influence of certain cheap chemicals therewith. Intimate mixing by harrows, plows, and sprinkler to apply chemicals. Date of beginning, July 1, 1922. Cooperating agencies: U. S. Bureau of Public Roads, University of Georgia Road Laboratory, and various county authorities of Georgia.

GEORGIA. *University of Georgia.* Top-soil roads.

Research on effective life of sand-clay, top-soil, and semi-gravel roads in Georgia on the State roads to determine periods of durability, cost of repairs, cost of rebuilding, residual value at reconstruction, and scientific examination of the sand, clay, and silt as to binding value and behavior in the presence of water. Standard sieves, centrifuge, ground glass discs for tenacity test of clay, and ordinary laboratory equipment. Also mechanical sifter. Details in the University of Georgia Bulletin, 1922. (Volume XXII No. 5a.) Continuation of research but with more careful study of clay and silt which has been in progress in the University of Georgia since 1907. Date of beginning, July 1, 1922. Cooperating agencies: U. S. Bureau of Public Roads and University of Georgia Roads Laboratory.

MINNESOTA. *University of Minnesota.* Value of marl as a binder in sand roads.

To determine the value of marl in the construction of highways. The work is carried on by a special appropriation by the state for the investigation of marl. Briquetting machine, impact machine, electric drying oven. The properties of marl and sand mixtures are being investigated by comparison with the similar properties of clay and sand. During the summer, short pieces of road bed will be constructed, using different mixtures of marl and sand, and the data will be obtained from these sample roads. Date of beginning, 1922.

NORTH CAROLINA. *State Highway Commission.* Wearing qualities of sand clay roads.

To attempt to determine where sand-clay road value stops and hard surface begins. Wear measurements, levels, and traffic census on sand clay roads under medium and heavy traffic. Cooperating agency: U. S. Bureau of Public Roads.

North Carolina. *State Highway Commission.* Surface treatment of sand-clay roads in preparation for bituminous surfacing.

To find a road surface of low cost that will be dustless and capable of carrying traffic more economically than earth roads and yet not cost the amount of a hard surface road nor be so heavily constructed. Roll a single layer of 2″ stone into a sand-clay or top-soil road after it has been scarified and after it has set up, cover this with cold bituminous material and later cover with sand and open to traffic. Date of beginning, December, 1921.

Texas. *Agricultural and Mechanical College.* Sand clay roads.

See Texas Engineering Experiment Station Bulletin No. 19.

Wyoming. *State Highway Department.* Sand-clay surfacing materials.

35.53 Top-soil and Sand-clay, Cross Section

District of Columbia. *U. S. Bureau of Public Roads.* Sand-clay and top-soil road investigations.

Study of sand-clay and top-soil roads with the following purposes in view: 1. To determine the service value (class, quantity and character of traffic sustained). 2. To determine the characteristics of materials best suited for this type of construction. 3. To improve the service value of roads of this type already constructed. 4. To determine maintenance costs and best methods. Stretches of road made up of several sections of the same type of construction, but subjected to different intensities of traffic selected in several states for study. Every effort made to select these stretches so that the maximum intensity of traffic in one section will be higher than sand-clay construction should preferably carry, the other sections carrying traffic within safe limits for sand-clay or top-soil. Traffic studies will be made according to a standard plan of traffic census enumeration. Careful records made of the history, design and construction of the section. Subgrade conditions noted and studied. Careful observations made of the change in configuration of the road surface, both transversely and longitudinally. Samples of the road surface sent to the laboratory and subjected to suitable physical tests for purpose of coordinating physical characteristics with the service behavior. The results of the investigation analyzed to determine the kind and intensity of traffic that can be economically carried by this type of construction. Date of beginning, July, 1921. Cooperating agencies: National Research Council, Association of State Highway Officials, State Highway Departments and Road Materials Committee of the A. S. T. M.

35.6 Dirt Roads

District of Columbia. *National Lime Association.* Use of lime for hardening soft spots in clay roads or sub-base.

Texas. *Agricultural and Mechanical College.* Earth roads.

See Texas Engineering Experiment Station Bulletin No. 1.

Wyoming. *State Highway Department.* Light construction with tractor blade.

36. STRUCTURES

NEW YORK. *American Concrete Institute.* New York City reinforced highway bridges and culverts.

Report of Committee Proceedings, 1922.

36.1 Bridges

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Bridge impact tests.

To determine impact stresses in various portions and various members of highway bridges under the action of different kinds of loading. A number of representative types of structures selected and enough instruments attached to give autographic records from which the deformations may be obtained. Loading varied from very light to very heavy trucks. The tires likewise varied, using pneumatic as well as solids. In every case very accurate measure of the roughness of the bridge floor taken. The test truck fitted with an autographic device for recording the actual impact on the bridge floor. The effect of speed noted. Date of beginning, 1921.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Investigation of skew arches.

To obtain information leading to the more rational design of skew arches. Load-applying devices, Berry strain gage, special overhead construction for applying loads. Model reinforced concrete arches, span about 7 feet. Angles of skew to vary. A large number of concentrated loads will be applied to resemble the condition of uniform loading and deformations of the arch will be measured. Date of beginning, July 1, 1922.

ILLINOIS. *Department of Public Works and Buildings, Division of Highways.* Deformations of concrete slab bridge due to temperature changes.

Measured by strainograph attached to pier.

ILLINOIS. *University of Illinois.* Movement of rock abutments.

The horizontal movement of the sides of a rocky ravine induced by change of temperature is measured.

IOWA. *State College.* Investigation of load distribution in highway bridge floors.

To determine the load concentration on the steel I-beams of I-beam-concrete slab highway bridges. Berry and Fuller strain gages, special highway bridge floor, jacks, etc. 1919–1921. Cooperating agency, Iowa State Highway Commission.

IOWA. *State College.* Load concentration on steel floor joists of wood floor highway bridges.

(See Bulletin No. 53, Eng. Exp. Sta.)

IOWA. *State College.* Impact on bridges.

To measure the effect of impact on various highway bridges. Strain gages, trucks and special apparatus. Date of beginning, 1921. Cooperating agency, U. S. Bureau of Public Roads.

MINNESOTA. *University of Minneosta.* Secondary stresses on Kenova Bridge.

To compare measured stresses with calculated secondary stresses in bridge. The secondary stresses were calculated from the design of the bridge and then checked with measurements taken on the completed bridge by means of strainographs. 1916–1922.

OKLAHOMA. *State Highway Department.* Determination of the great loss of bridges over the South Canadian River.

PENNSYLVANIA. *Lafayette College.* Impact stresses in highway bridges and truck loads.

This will be undertaken and completed this summer in connection with committee XV, American Railway Engineering Association, to determine the proper impact coefficient to use in the design of highway bridges. Turneaure extensometers and Frankel deflectometer and trucks and bridges of various capacities and types. July 5, 1922–January 1, 1923. Cooperating agency, American Railway Engineering Association.

WEST VIRGINIA. *University of West Virginia.* Determination of the stresses in highway bridge floors.

Not begun, August 3, 1922.

37. EXPERIMENTAL ROADS

37.1 California, Pittsburg

CALIFORNIA. *Columbia Steel Company,* . Pittsburg, California, test road.

Constructed of 1-2-4 cement concrete, May, 1921. Oval track one-fourth mile in length on subgrade of adobe soil which was built up in 6″ layers, thoroughly pulverized and rolled. Thirteen sections of 18 foot pavement. *Variables:* Thickness: 5, 6, 7, and 8 inches; Shape: solid uniform thickness, solid 6″ center tapering to 9″ at edge, cellular with edge or edge and center depressed ribs, 7″, 8″, and 12″ deep; Reinforcement: structural and hard grade open hearth billet steel, 20–69 tons per mile, placed in center and in top and bottom of slab. With and without longitudinal joints. Four tunnels built transversely under road permitted observations of deflections of slab and subgrade under static, moving, and impact loads. Deformation due to temperature and subgrade moisture conditions were studied. Extensometers measured surface strains. Traffic test was begun November 9, 1921, with nineteen 3½-ton Standard trucks and sixteen 5-ton Pierce-Arrow trucks, each loaded to gross weight of 14,500 lbs. Final loading of Pierce-Arrow trucks 24,000 lbs. Several Liberty trucks loaded to gross weight of 27,000 lbs. were added December 29, 1921. Traffic was suspended on January 28, 1922, 3,668,100 tons of traffic having passed over the highway at the rate of 10,000 trucks passing a section in one day at a speed of approximately thirteen miles per hour. Traffic resumed June 1. In August, a two-wheel trailer, loaded to 100,000 lbs. gross, or 5,000 lbs. per inch width of 10-inch steel tires, was operated three times around the circuit on the outside of the track. See Engineering News-Record, June 29, 1922.

California State Highway Commission, Pittsburg Test Road, Pittsburg, California.

To investigate strength of various designs of highway sections, cement concrete and bituminous. A continuation of the test of the Pittsburg test road begun by the Columbia Steel Company. Ten 90 foot sections of pavement will be laid on the tangent of the former test road, two sections of which will be asphaltic concrete. Longitudinal as well as transverse tunnels will be used, and also tunnels under transverse joints. Concrete sections on the curves of the former road which were not broken will be used in this third service test. Cooperating agencies: U. S. Bureau of Public Roads and Columbia Steel Company in the use of its property.

37.2 Delaware, New Castle County

DELAWARE. *Delaware State Highway Department.* Philadelphia Pike, New Castle County, Delaware.

Constructed September, 1921. 1,200 feet long in eight sections, 20 feet wide. Center construction joint. *Variables:* Reinforcing: single layer of rods in top or bottom, or wire reinforcing in top. Mix: 1-2-4; 1-½-3. Aggregates: crushed stone, pebbles, slag. Admixtures: Cal, lime. Subgrade: crowned, flat. Alignment: level, 4% grade.

37.3 Illinois, Bates

ILLINOIS. *Department of Public Works and Buildings.* Bates experimental road, near Springfield, Illinois.

Constructed 1921. Tangent two miles long, practically level, 18 feet wide, on typical brown silt of corn belt of Illinois. To determine the resistance of various pavements subjected to increasing loads. Investigations also deal with subgrade actions, temperature deformation, etc. Two miles of road consisting of 75 separate and distinct sections of road, each section varying in length from 100 feet and constructed of a different specification to determine the relative merits of each. Types represented are brick of several constructions, asphaltic concrete, and Portland cement concrete. Various forms of joints and methods of reinforcing are represented. Tests of strength of slab corners under repeated static loads. Traffic test begun March 30, 1922, with 3-ton Liberty trucks with rear wheel load of 2,500 lbs. Tests under way August, 1922. For full description of pavement, see Engineering News-Record, August 18, 1921, and February 1, 1922. Cooperating agency: U. S. Bureau of Public Roads.

37.4 New Jersey, Warren County

NEW JERSEY. *State Highway Commission.* Experimental Portland cement concrete road between Philipsburg and Washington, Warren County, New Jersey.

Built in 1912. 4,800 feet in length of which 1,200 feet is 14 feet wide, 700 feet is 18 feet wide, and 2,900 feet is 16 feet wide. Concrete is proportioned 1-1 ½-3, 1-2 ½-5, and ½-4. Slab laid on old macadam 14 feet wide. In places, slab is 4″ to 5″ thick at center and 7″ thick at sides.

37.5 Pennsylvania, Byberry-Bensalem

PENNSYLVANIA. *Philadelphia City Bureau of Highways.* Byberry-Bensalem Pike.

Service test road 18,000 feet long comprised of 26 pavement sections including Portland cement concrete, vitrified block and bituminous types. Constructed 1912–13. Final report 1920. See Annual Report of the Bureau of Highways of the City of Philadelphia, Dec. 31, 1920.

37.6 Pennsylvania, Lancaster

PENNSYLVANIA. *State Highway Department.* Experimental road 6,000 feet long at Rheems, Lancaster County, Pennsylvania, containing various mixtures of cement concrete with various aggregates and finishes.

Constructed 1920 and 1921. To determine the relative value of various mixes of cement concrete and different coarse and fine aggregates as a wearing surface for cement concrete roads. In this road the same section and reinforcing is used throughout, the only difference in the various sections being the mix and aggregate used and also the method of finishing the surface. *Variables:* Fine aggregate: crushed sandstone, combination of limestone grit, and Susquehanna River bar sand, Delaware River sand. Coarse aggregate: limestones, trap rock sandstones, "heavy" slag, "light" slag, crushed gravel. Finish: machine, hand. Periodical observations of service test road to determine the effect of traffic and weather.

37.7 Texas, Neuces County. Shell concrete.

The road is a part of the Corpus-Robstown Road and has been constructed about four months. The section, 16 feet wide, 6″ thick on edges, 7″ in center and is 2,600 feet long. The mixture was 1 part cement to 4½ parts beach run sand, and was run in 1-sack batch mixer very wet. There was no tamping done and the pavement was finished with an ordinary sidewalk float. The shell used was a clean, live grade of shell ranging in size from ¼″ to 1½″ and contained about 33% sand. The top was ½″ trap rock with a mixture of tar and asphalt. The road was cured with wet dirt for a period of 18 days. There are two 40-foot sections of shell concrete in the concrete road between Corpus and the causeway that were built to replace sections washed out during the last storm. One section was open to traffic about a year before a bituminous topping was put on. The top of this section is of the same material as that on Robstown Road.

37.8 Virginia, Alexandria County

VIRGINIA. *U. S. Bureau of Public Roads, Alexandria County, Virginia.* Columbia Pike. Experimental concrete highway.

Constructed October, 1921; two miles long. Sections of various shapes, thicknesses, and reinforcements. Normal farm traffic. Tangent, curves, and grades. Sections of uniform thickness, and with inverted curbs.

37.81 Virginia, Arlington

VIRGINIA. *U. S. Bureau of Public Roads.* Experimental track at Arlington, Virginia. See 35.355.

37.9 Wisconsin, Milwaukee

WISCONSIN. *State Highway Commission.* Experimental road near Milwaukee. See 13.122.

To compare plain and reinforced concrete slabs with reference to cracks. Reinforcement in amount.

37.10 Wyoming, Casper

WYOMING. *State Highway Commission.* Reinforced concrete precast slab road in desert at Casper, Wyoming.

Constructed 1920. 2,400 feet long. Heavy truck traffic. To determine the practicability of constructing a cement concrete road consisting of precast slabs under conditions unfavorable for monolithic construction. Units are eight feet long, nine feet wide, and six inches thick. Some joints between slabs are of warped shape to hold surface in alignment; some diagonal. So far as is known, this is the only example of precast slab pavement construction on a sufficiently large scale to determine whether this method is successful on a road carrying heavy traffic. Cooperating agency, U. S. Bureau of Public Roads.

38. DESIGN OF SLAB

38.2 Impact Tests

DISTRICT OF COLUMBIA. *U. S. Bureau of Standards,* Impact tests of reinforced concrete slabs.

200 lb. cast iron ball dropped on slabs. Deflection and vibration measured by recording drum. Completed 1918. Cooperating Agency, Lehigh University. Report on file U. S. Bureau of Standards but not published.

40. DESIGN (VEHICLE)

(Census not developed. See Society of Automotive Engineers fo his division.)

41. DESIGN OF VEHICLE

CONNECTICUT. *Yale Sheffield Scientific School.* Friction losses in automobile chassis, involving solid and pneumatic rubber tires.

See Society of Automotive Engineers, Meeting of Metropolitan Section, April 23, to be published in Journal of that Society, fall, 1922.

CONNECTICUT. *Yale Sheffield Scientific School.* Investigation of shock-absorbing devices to determine their effectiveness for reducing body vibration.

To obtain a graphic record of body vibration produced by contact of the wheels with obstacles on a smooth road, to observe the cushioning action of the tires, springs, and special devices such as shock absorbers. Dynamometer drums with the surface fitted with one or more attached obstructions which will give a lifting and dropping motion to the wheels. A revolving strip of paper with instruments to give a record of vibration on the time base.

41.4 TIRES

41.41 Solid

NEW YORK. *Kelly-Springfield Tire Company, New York City.* Effect of solid and pneumatic tires on soft subgrade.

To determine area of contact of cushion tires, various sizes, under given loads and the corresponding areas of pneumatic truck tires of the same capacity. Pressure testing machines or loadometers, carbon paper, and plainmeter. Static loads applied measured, and area of impression of tire on carbon paper obtained by means of planimeter. August, 1921–December, 1921. Cooperating agency, laboratory and factory equipment of Cumberland Plant of Kelly-Springfield Tire Company.

42.13 Wear on Tires

INDIANA. *Purdue University.* Test of tires in road tests on various types of pavements.

43. ALIGNMENT

43.1 Curves

ILLINOIS. *Portland Cement Association, Chicago.* Superelevation of curves in relation to speed of vehicle.

Theoretical discussion and tests. See Proceedings of American Concrete Institute, 1918.

45. SAFETY

OHIO. *State University.* Effect of form of commercial reflectors on light distribution from different types of automobile headlighting lenses. Tests completed but not yet published.

TEXAS. *Agricultural and Mechanical College.* Glare from automobile headlights.

(a) To determine limits of light distribution from automobile headlights within which glare is not objectionable. (b) To determine distribution curves for principal types of reflectors and lenses. Simple lenses, photometric equipment, etc. Both laboratory and road tests are to be made. Date of beginning, March, 1922. Manufacturers have cooperated.

50. CONSTRUCTION

DISTRICT OF COLUMBIA. *Associated General Contractors of America.* Statistics on various subjects affecting construction.

Appear weekly in Index.

52.1 Concrete

52.13 Mixing

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Investigation of central plant mixed concrete.

To determine the safe maximum time of haul of central plant mixed concrete. Crushing strength specimens of concrete taken from batches of concrete mixed at a central plant and hauled in a motor truck for various periods of time. Various mixes used. The motor truck partitioned off so that the concrete for each lot of specimens may be taken without disturbing the adjoining concrete. Date of beginning, 1921.

INDIANA. *Purdue University.* Test of Koehring concrete mixer.

To determine optimum conditions of operation of mixer to produce concrete of desired strength and consistency. A 10-cubic foot paving mixer was operated electrically and the input of electric power measured at various stages of mixing. Time record of operations taken. Gravel and limestone aggregate. Effect of time of mixing and speed of rotation on strength and consistency of 1-2-3 concrete. Contents of drum discharged at specified time. Water well controlled in time and amount, and found to be important. See Proceedings of American Concrete Institute, 1921. Cooperating agency, Koehring Machine Company.

ILLINOIS. *Structural Materials Research Laboratory, Lewis Institute, Chicago.* Effect of time of mixing on strength of concrete.

See Proceedings American Concrete Institute, 1918.

52.14 Placing

OHIO. *Municipal University of Akron.* Regaging or remixing (cement) slag-concrete.

To determine whether concrete which has been mixed for some time may be safely used under any circumstances. Mixed batch of concrete. Molded 6"x12" cylinders at hourly intervals. Tried (a) mixing without adding additional water; (b) mixing wet, uniform consistency; (c) same as (a) except we did not disturb balance of pile, merely cut off enough for sample; (d) same as (c) except additional water was added to each cylinder. Date of beginning, April 1, 1922.

OHIO. *Municipal University of Akron.* Regaging or remixing mortar.

To ascertain whether mortar may safely be used after having been mixed for some time. Usual laboratory equipment. Mixed batch of 1:3 mortar (standard sand) formed briquettes at hourly intervals. Mixed part of it dry and added water to part. Reported in Engineering News-Record, November 10, 1921. April 1, 1921–June 1, 1921.

WISCONSIN. *State Highway Commission.* Effect of delay in placing on the abrasive resistance and strength of concrete.

To estimate the length of time which may elapse between the mixing and the placing of the concrete in road construction, using central mixing plant. Thirty-two slabs and forty-eight cylinders were made of 1:2:3½ proportions by volume of cement, sand, and stone. These specimens varied in regard to placing (periods of delay in placing were used 0, 1, 2, 3 and 4 hours) and curing. Conclusions were obtained from compression and abrasion tests. Completed July 15, 1920.

52.17 Curing

ILLINOIS. *Department of Public Works and Buildings.* Curing of concrete at low and usual temperatures.

Transverse and compression tests on specimens treated with 2% flake calcium chloride, 2% granulated calcium chloride, 2½% flaked calcium chloride, dry flake calcium chloride, three lb. to square yard of surface. Twenty-three different methods of curing with nine agents were investigated at usual temperatures. Reported in Engineering News-Record March 9, 1922.

NEW YORK. *Commission of Highways.* Curing concrete at low temperatures.

Results in compression on 6-inch cubes 1-1 ½-3 mix, some with calcium chloride solution, cured both outdoors and indoors under temperatures with extreme ranges of minus 2 degrees and 44 Fahrenheit, at ages of 3, 7, 10, 14 and 28 days. Fine aggregate, iron ore tailings, coarse aggregate, limestone.

PENNSYLVANIA. *State College.* To determine when the form should be removed from a concrete wall.

Concrete specimens, apparatus for measuring electrical resistance. To measure electrical resistance of specimens of given mixture from time to time with weight and other data to determine the relation, if any, between electrical resistance and strength. Date of beginning, June, 1921.

52.18 Resurfacing

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Investigation of methods of resurfacing of old concrete pavements.

Inspection of the various methods for re-surfacing old concrete to be made and results compared.

WISCONSIN. *Milwaukee County.* Resurfacing concrete pavements.

52.5 Earth and Sand-clay

52.52 Placing

WISCONSIN. *Department of Highways and Streets, Beloit.* Methods of settling trenches in sand and gravel soil.

To settle fresh trenches so that no appreciable settlement will take place under pavement laid over them within a month. Ordinary tools; also 6 foot length of 2½ inch G. I. pipe with standard fire hose coupling on one end. Hand tamping, hand flushing, while back filling is being done (filling into water); 1916–1918. Co-operating agency, Beloit, Wisconsin, Water, Gas, and Electric Company.

55 COST ACCOUNTING

DISTRICT OF COLUMBIA. *Associated General Contractors of America.* Uniform cost accounting for highway construction.

To devise a uniform cost accounting system for highway construction so that the work may be scientifically estimated and accounted. Date of beginning, January, 1922.

DISTRICT OF COLUMBIA. *Associated General Contractors of America.* Equipment costs.

To ascertain the cost of owning and operating various construction machines and to devise a method for estimating the depreciation and other costs with reasonable accuracy. Date of beginning, April, 1920. Cooperating agencies were various manufacturers and rebuilders of equipment.

55.1 Standard Estimating and Cost

District of Columbia. *Associated General Contractors of America.* Standard estimating guide sheet for highway construction.

To ascertain the items of cost ordinarily incurred in highway work and to systematize the items for use in estimating and cost accounting. Date of beginning, January, 1922.

56 CONSTRUCTION CONTRACTS

District of Columbia. *Associated General Contractors of America.* Cost of contingencies under present highway contracts.

To show how the cost of highway construction is increased by introducing in the contract responsibilities for many contingencies which can be more cheaply assumed by the State itself. Cooperating agency, American Association of State Highway Officials. Date of beginning, January, 1922.

District of Columbia. *Associated General Contractors of America.* Economy of awarding contracts in advance of the construction season.

To show the saving that can be effected by awarding contracts well in advance of the construction season and to show a means of reducing seasonal unemployment. See Bulletin of above society for August, 1921, page 23. Date of beginning, January, 1920. Completed June, 1921.

57 CONSTRUCTION ECONOMICS

District of Columbia. *Associated General Contractors of America.* The most economical methods of highway construction.

To ascertain the general methods of construction most economical under certain field conditions. Date of beginning, January, 1922.

60. MATERIALS

District of Columbia. *U. S. Geological Survey.* Collection, collation, dissemination of information on the geology and mineral resources and industry of foreign countries.

The index files to this information now number about 35,000 cards each; one file is arranged by subjects and the other by countries. These reference files are open to consultation at any time.

61. AGGREGATES

Kentucky. *University of Kentucky.* Concrete aggregates.

Fine gravel and sand from the lower Ohio River, sandstone with very fine sand from eastern Kentucky. Crushing strength as compared with physical properties of stone.

New York. *Columbia University.* Tests on concrete (material tests).

OHIO. *National Slag Association, Cleveland.* Tests on two hundred aggregates.

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Tests being conducted by Prof. Duff A. Abrams.

61.1 Fine Aggregates

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Concrete-making properties of fine aggregates.

61.11 Sand

61.111 Production

61.1114 Preparation and Treatment

NORTH CAROLINA. *State Highway Commission.* Effect of different cements on different sands.

To make use of sands showing by the present tests to be of inferior quality.

TENNESSEE. *State Highway Department.* Treatment of local low-grade sands to bring them up to a standard.

Organic matter chief depreciatory factor. Investigation has been completed on local low-grade sands on one project only. The trouble with this sand was a uniformity in the size of the sand particles and the problem was to obtain proper gradings and tensile strength tests. Cooperating agency, University of Tennessee.

NORTH CAROLINA. *State Highway Commission.* Accelerators.

See report of Committee 69, American Society for Testing Materials.

PENNSYLVANIA. *State Highway Department.* Tests of accelerators.

To determine the relative value of different chemical accelerators 1921–1922.

61.112 Tests

CONNECTICUT. *Hartford City Laboratory.* Fineness modulus of sands.

To study the local sands as to their desirability for concrete road construction. The fineness modulus of standard Ottawa sand assumed to be 2.65. The fineness modulus of the sand in question is determined. A theoretical tensile strength yield is then found by dividing the fineness modulus of the local sand by the fineness modulus of the standard Ottawa sand. Date of beginning, 1918.

DELAWARE. *State Highway Department.* Investigation of local sands.

GEORGIA. *School of Technology.* Effect of material below 80 mesh in sand on strength and surface finishing of concrete.

In using coarse sands in concrete work, it has been impossible to finish same, but on addition of fines, the finish becomes smoother. Actual investigation on concrete as it is being poured in forms. Tensile strength machine for mortar strength. Date of beginning, June 1, 1921. Cooperating agency, R. C. Campbell Sand Company.

GEORGIA. *School of Technology.* Effect of proportion of cement to sand on the mortar strength ratio.

To determine the cement that must be used with different sands to produce a 100% strength ratio, in tension and compression. May 1, 1921–December 1921. Cooperating agency, Chehaw Sand and Gravel Co., Atlanta, Georgia.

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Tests of light-weight aggregates.

MAINE. *University of Maine.* Amount of cement necessary for Maine sands to give certain strength in tension and compression.

To classify types of fine aggregate as to their relative desirability for use in concrete. Date of beginning, November 1920. Cooperating agency: Maine State Highway Commission.

PENNSYLVANIA. *Department of Highways and Streets, Altoona.* Tests on Pennsylvania sands.

WISCONSIN. *State Highway Commission.* Report of tests of concrete made with screenings for fine aggregates.

To determine the value of screenings as fine aggregate for concrete to be used for road construction. Variables: 1. Coarse aggregates: crushed stone, of three degrees of hardness. 2. Specially prepared mixtures of screenings which varied considerably in dust content from each of these crushed stones. 3. Fine aggregates: consisting of mixtures of screenings and fine, clean sand with the proportion of sand predominating in the mixture. 4. Fine aggregates: consisting of mixtures of screenings and well-graded sands. A few tests were run on the effect of time of mixing on the strength and wear resistance of concrete, using screening with a high percent of dust for fine aggregate. Smith Six Cubic Foot Mixer. Results based on tests of one hundred and fourteen 4⅛x8x19½ inch slabs for abrasion, seventy-eight 6x12 inch cylinders, and 66 mortar briquettes. Completed February 23, 1921.

WISCONSIN. *State Highway Commission.* Report of tests on concrete made with sandstone sand and tailings for fine aggregates.

To ascertain the possibility of using sands made by rolling or crushing poorly cemented St. Peters sandstone (southwestern counties of Wisconsin), either alone or in combination with screenings or mine tailings for fine aggregate in concrete road construction. Specimens made of different proportions and subjected to tests in compression and rattler abrasion test. Completed March 21, 1921.

61.1123 Methods of Test

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Crushing tests of concrete aggregate; abrasion tests of aggregate.

MAINE. *University of Maine.* Relation between fourteen and twenty-eight day test on mortars. Tension and compression.

To determine if the fourteen day test is more reliable for preliminary acceptance of sands than the twenty-eight day test. Date of beginning, January 1922. Cooperating agency, Maine State Highway Commission.

NORTH CAROLINA. *State Highway Commission.* Comparison of the relation of tension tests of fine aggregates in mortars to the compressive strength in concrete.

PENNSYLVANIA. *Lafayette College, Easton.* To determine when fine aggregates proportioned on Abrams' design theory will meet twenty-eight day strength requirements.

To determine a test at an earlier period than twenty-eight days for fine aggregates which are proportioned by Abrams' theory to give certain strength of twenty-eight days. May 1, 1922–June 1, 1922.

PENNSYLVANIA. *State Highway Department.* Tension and compression tests of mortar.

To obtain comparative results between tension and compression tests of standard Ottawa sand and natural sand mortar. Date of beginning, 1921.

PENNSYLVANIA. *State Highway Department.* Comparative relation in strength of cement-sand mortars when proportioned by volume and weight.

Ottawa and various other sands, mixed 1:2, and 1:3 by weight and volume and tested in tension and compression. Completed 1920.

TENNESSEE. *University of Tennessee.* Tests of Tennessee sands as a fine aggregate for concrete.

To determine relative strengths of concretes made from Tennessee sands and the value of the Ottawa sand test as a criterion for the value of sand. Cooperating agencies: Tennessee Highway Department and University of Tennessee.

61.1124 Standard Tests

GEORGIA. *School of Technology.* Determination of strength-ratio of sands for other times of set than twenty-eight days.

To find whether a sand mortar has the same strength-ratio to standard sand in tension at age of seven days, twenty-eight days, six months, one year, and three years. November 1, 1921–November 1, 1924.

61.1125 Fundamental Properties

MISSOURI. *University of Missouri.* Physical properties of Missouri sands.

NEW HAMPSHIRE. *State Highway Department.* Study of New Hampshire sands to be used in concrete.

To determine, if possible, whether or not granite sands show disintegration in concrete over a period of two years. May 1, 1922–May 1, 1924. Cooperating agencies: U. S. Bureau of Public Roads.

OHIO. *State Highway Department.* Apparent specific gravity of fine aggregates.

Determination of the apparent specific gravity of fine aggregates. 500 cc. glass stoppered cylinder. 500 grams of dried clean sand placed in cylinder; pour in about 20 cc., or sufficient kerosene to saturate sand grains; shake; introduce 250 cc. of water; shake vigorously to bring excess kerosene to top; read water level; from volume displaced compute apparent specific gravity. Completed 1917.

PENNSYLVANIA. *State Highway Department.* Wear tests on mortars containing different sands.

To determine for sands the relation between strength and resistance to wear under abrasion when mixed in mortars of the same. proportion. Dorry hardness machine with special holders for inserting 2″ by 4″ mortar cylinders under definite loads. 20,000 lb. compression machine. Completed 1921.

61.114 Impurities

CALIFORNIA. *Leland Stanford University.* Silt investigations of various counties.

COLORADO. *Agricultural College.* Effect of washing sands containing excess silt.

CONNECTICUT. *Hartford City Laboratory.* Colorimetric tests on sand.

Effect of organic impurities on our local sands. Date of beginning, 1918.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Method for detecting and neutralizing deleterious substances in concrete aggregates.

A. Detecting deleterious substances in aggregates. To develop a simple method for detecting substances harmful to cement mixtures. Study of those sands which give low strengths with no apparent reason, i.e., sands of good quality and grading. B. Neutralizing deleterious substances in aggregates. To develop methods for neutralizing the harmful effect of the common deleterious substances such as organic matter and alkalin salts when found in concrete aggregates.

PENNSYLVANIA. *State Highway Department.* To determine the effect of coal on the strength of sands.

To determine the amount of anthracite or bituminous coal in sand which is harmful for use in concrete roads. Powdered anthracite and bituminous coal in varying proportions added to natural sands for tension and compression tests. Completed 1921.

PENNSYLVANIA. *State Highway Department.* Determination of clay and silt in concrete sands.

To obtain comparative results between volumetric and gravimetric methods of determining loam in natural sand so that the field tests are comparable to laboratory results. Date of beginning, 1921.

VIRGINIA. *Polytechnic Institute.* The effect of organic matter in sand.

VIRGINIA. *State Highway Commission.* Organic matter in sand.

To determine if the cause of failure of sands containing organic matter is of a chemical or physical nature. Chemical treatment and comparative mortar strength tests of treated and untreated sand. Begun in 1916 by Mr. Shreve Clark at Columbus, Ohio, in Ohio State Highway Testing Laboratory.

WISCONSIN. *State Highway Commission.* A tentative adopted plan for making silt tests.

Two glass bottles, jars, or graduates, which have uniform bore over a depth of 8″. The minimum diameter should not be less than 1½ inches. Fill vessels 2½″ with two representative samples respectively. Add water to make total depth 5″. Shake for 30 seconds, allow to stand for one hour. Read depth of silt to nearest 1/100″. Read depths of sand and silt making four measurements at different points. If percent of silt exceeds the 8% standard, allow to stand for 4 hours and read again. If percent still exceeds the standard, reject, or send 25 lb. sample to testing laboratory. Completed January 1, 1922.

61.2 Coarse Aggregates

61.21 Stone

61.212 Tests

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Study of the wear-resisting properties of concrete aggregates.

To determine effect of variations in quality and size of coarse aggregates on the resistance to wear and strength of concrete used in road construction. Information sought on (a) relations existing between the wear and strength of concrete, (b) relation existing between the wear of concrete specimens and wear of aggregates used, and (c) the relation existing between the wear and strength of concrete specimens in which the size of the coarse aggregate varies. To determine the effect of variations in the type, quality and grading of fine aggregate on the wear and strength of concrete used in road construction. Information is sought on (a) relation existing between the wear and strength of concrete in which different types of sand, stone and slag screenings are used as fine aggregates, (b) whether the usual laboratory tests for fine aggregates furnish a reliable index of the resistance to wear and crushing strength as determined by this investigation and (c) the influence of mechanical analysis of the fine aggregate on the wear and strength of concrete. Date of beginning, July 1, 1921.

GEORGIA. *School of Technology.* The relation between strength-ratio of sand, abrasion tests of stone, and strength of concrete.

Slag coarse aggregate moulded with sands of different strength-ratios and twenty-eight day compressive strength on 6" by 12" cylinders obtained. Aggregates of different abrasion losses (slag-stone-gravel) were used with the same sand and compressive strength obtained as before. Date of beginning, January, 1920.

ILLINOIS. *Department of Public Works and Buildings.* Determination of effect of novaculite as coarse aggregate on the transverse strength of concrete.

To determine if novaculite can be used as a coarse aggregate in concrete with no resulting decrease in transverse strength over concrete containing limestone aggregate. December 21, 1921–March 21, 1922.

ILLINOIS. *Department of Public Works and Buildings.* Determination of the effect of hardness of stone on the transverse strength of concrete.

To determine if the hardness of rock, indicated by its French coefficient, has any relation to the compressive and transverse strength of concrete. Tests at thirty days and sixty days. French coefficient of soft stone averaged less than half of those of the hard stone. Absorption of soft stone, 2%; of hard stone, 0.5%. January 17, 1922–March 30, 1922.

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Concrete-making properties of large-sized aggregates.

Compression tests on 8x16" concrete cylinders using pebbles, crushed limestone, trap, granite, and slag up to 4" in size.

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Concrete-making properties of coarse aggregates.

MINNESOTA. *University of Minnesota.* The effect of weathered and unweathered feldspar on concrete mortars.

To determine what effect feldspar in a weathered and an unweathered condition will have on Portland cement mortars. Several series of briquettes have been made, using sand of a given grading and containing different amounts of feldspar, both weathered and unweathered. Briquettes are tested at the age of seven and twenty-eight days. June, 1921–June, 1922. Cooperating agency, Geological Department, University of Minnesota.

MINNESOTA. *University of Minnesota.* Suitability of certain limestones and sandstones of the state for use in concrete pavements.

To determine whether some of the rocks found and quarried in the State which fail to meet some of the present requirements of the Minnesota Highway Department, but which in many cases can be produced and delivered to the job at a lower cost than some of the better rocks, should be used in concrete pavement. Concrete cylinders (5x12) were made of Fowler and Pay limestone, Minnesota Crushed Stone Company limestone, Kettle River Sandstone, and Dresser Junction trap, using a 1:2:4 mix. One-third of these cylinders were subjected to a condition of alternate freezing and thawing, one-third were given laboratory conditions. These cylinders will be broken at the age of six months. The other third have been placed out in the open, where they will be exposed to varying weather conditions. These cylinders will be broken at the age of one year. Begun December, 1921. Cooperating agencies: Illinois and Iowa State Highway Departments.

NEBRASKA. *Omaha Testing Laboratory.* Experiments on stone and sand-gravel to determine their relative value as aggregate for concrete.

NEW YORK. *Cornell University,* Washed culm as aggregate for concrete.

Date of beginning, September, 1921.

TENNESSEE. *University of Tennessee.* Tests of mascot chats as a concreting material.

To determine comparative strength and increase in strength with age, as well as effect of freezing on concrete. Ordinary compression cylinders, cured at laboratory temperature, and broken in ordinary way. Freezing accomplished by placing in 200 lb. blocks of ice and frozen with ice in plant. September, 1921–June, 1922. Cooperating agencies: Holston Quarries, Kinzel Thompson Sand and Gravel Co.

WASHINGTON. *City of Seattle, Engineering Department Laboratory.* Test for strength and wearing quality of sandstone concrete for pavements.

To find out if concrete pavement made of sandstone aggregate could be used on grades to stop skidding in place of sandstone blocks. Slabs for transverse test 17"x48" with varying thickness, 6"x12" concrete cylinders. January 19, 1922–April 10, 1922.

61.2123 Methods of Tests for Stone Aggregates

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Standardization of a crushing strength test for rock.

To determine the most accurate method of obtaining the resistance to compression of stone used as concrete aggregate and in stone block pavement construction. Determine the effect of the following variables on the crushing strength: 1. Rate of speed of application of load. 2. Shape and dimensions of the test pieces. 3. The condition of the bearing surface of the test piece. Examination of the effect of each factor. Date of beginning, July 1, 1921.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Study of Laboratory tests for determining quality of aggregates used in concrete.

To develop laboratory tests for determining the quality of aggregates for use in concrete, viz., stone, gravel and sand.

ILLINOIS. *Department of Public Works and Buildings.* Comparison of slotted and deval plain cylinders for abrasion tests of stone.

To determine if the results obtained with plain cylinders differ materially with slotted cylinders in the abrasion tests for stone. January 7, 1922–February 20, 1922. Cooperating agency, Committee on Tests and Investigations, American Association of State Highway Officials.

INDIANA. *State Highway Commission.* Adoption of abrasion test for stone chips as delivered.

IOWA. *State Highway Commission.* Tests of stone used in aggregates for concrete paving.

To determine those physical characteristics of Iowa limestone that vitally affect the quality of concrete, and to devise tests that will consistently indicate variations of these characteristics. Deval machine, with Standard and slotted cylinders. Date of beginning, December, 1921.

IOWA. *State Highway Commission.* Durability of limestone aggregates.

To determine a practical test for the durability of limestone in pavements. Accelerated weathering tests, freezing and thawing, chemical analysis, and soundness tests. Date of beginning, April, 1922.

MASSACHUSETTS. *Institute of Technology.* A study of possible improvements in the Deval abrasion test for road materials.

Reported in a thesis, June, 1922, on file in Civil Engineering Library, Massachusetts Institute of Technology, 1921–1922.

MICHIGAN. *University of Michigan, State Highway Laboratory.* Comparison of Deval and slotted clyinders for abrasion tests on rock.

To determine comparative losses. Completed May, 1922.

NEW YORK. *Cornell University.* Strength of rock.

To prepare tests for quality and specifications for standard tests. Date of beginning, September, 1921. Cooperating agency, Committee D-4, American Society for Testing Materials.

PENNSYLVANIA. *Lafayette College.* Effect of variations from those required by specifications in the size of screen openings and effect of location of samples and method of sampling of stone.

To determine, if possible, the allowable variation in size of screen opening which may be permitted without materially affecting strength concrete. Specimens will be taken from screens of different makes and angles and samples will be taken from different car loads and from different parts of the same car loads when car loads have and have not traveled considerable distances. March 12, 1922–September 15, 1923. Cooperating agency, General Crushed Stone Co., Easton, Pennsylvania.

WISCONSIN. *State Highway Commission.* Results of abrasion tests on crushed stone compared with those on gravel.

To secure information concerning the results from abrasive tests made on crushed rock in the sizes actually used for concrete aggregate. The tests were made in Deval cylinders of plain and slotted type on six samples of crushed stone and four samples of gravel. Each aggregate was graded in two sizes, material passing a 2" screen and held on a 1" screen, and material passing a 1" screen and held on a ½" screen. Completed November 3, 1920.

61.221 Production—Gravel

INDIANA. *Purdue University.* Study of gravel aggregates to determine economy of production, in relation to technical properties of gravel for concrete.

Cooperating agencies: U. S. Bureau of Public Roads; Indiana Sand and Gravel Producers Association.

61.2214 Preparation and Treatment

INDIANA. *Sand and Gravel Producers Association, Indianapolis.* Methods of packing or consolidating gravel.

61.222 Tests of Gravel

IOWA. *State Highway Commission.* Effect of shale pebbles in aggregates on the quality of concrete.

Date of beginning, December, 1921. Cooperating Agency, Minnesota State Highway Department.

MINNESOTA. *University of Minnesota.* Effect of shale pebbles in concrete and removal of shale from gravel.

To determine the effect of shale pebbles on the compressive strength of concrete and to devise laboratory and commercial methods for removing shale from gravel. Forms for concrete cylinders, flotation apparatus for removing shale from gravel, 200,000 lb. compression machine. A number of tests have been made on concrete containing different amounts of shale and subjected to alternate freezing and thawing conditions. Methods of separating shale from pebbles have been investigated. March, 1922.

61.2223 Methods of Testing Aggregates, Gravel

INDIANA. *State Highway Commission.* Development of abrasion test for gravels.

INDIANA. *Purdue University.* Abrasion tests on road materials.

To correlate the wearing qualities of stone and gravel. 1. Determine the French coefficient of stone under standard test. 2. Wear crushed stone down to size and shape of gravel. 3. Run tests on (a) stone-equal parts of ½-¾, ¾-1, 1-1½, .1 ½-2, (b) gravel-equal parts of ½-¾. ¾-1, 1-1½. 4. Make synthetic mixtures of various sizes of graded stone and gravel and test as in above 3. Date of beginning, March, 1922.

OHIO. *State Highway Department.* Abrasion test for gravel.

Design of wear test for gravel aggregates to be used in concrete wearing course or gravel roads. Deval cylinder—6 cast iron spheres, 1.875-inches in diameter, weighing 95 lbs. each as specified by the American Society for Testing Materials for standard rattler test for paving brick. Gravel screened through the following circular screens: 2″-1½″-1″-¾″-½″. After drying, the following weight taken: 1250 grams each 2″ -1½″, 1½″-1″, 1″-¾″, ¾″-½″, total 5,000 grams. Speed, duration of test, etc., same as standard Deval test for stone. See Bulletin 949, U. S. Bureau of Public Roads. Completed 1914.

PENNSYLVANIA. *State Highway Department.* Abrasion tests in standard Deval and slotted cylinders.

To develop a better test for gravel and the softer sandstone and limestone than the standard Deval test. Date of beginning, 1921. Cooperating agencies: American Association of State Highway Officials, U. S. Bureau of Public Roads, and the Lewis Institute.

VIRGINIA. *State Highway Commission.* Comparison of losses in Deval cylinder closed and with lid reversed and raised 1/16 of an inch.

61.2225 Fundamental Properties

MISSOURI. *University of Missouri.* Physical properties of Missouri gravel.

NEBRASKA. *Department of Public Works.* Platte river sand-gravel aggregate.

NORTH DAKOTA. *University of North Dakota.* Lithology of the gravels of North Dakota.

To determine the kind of mineral fragments composing the gravels. Pebble counts and microscope. Taking representative samples and identifying the pebbles. Date of beginning, June, 1921. Cooperating agency, North Dakota State Highway Commission.

61.223 Specifications—Gravel

INDIANA. *Purdue University.* Tolerance of coarse aggregate passing the one-quarter inch sieve as affecting specifications for gravel aggregate.

To determine to what extent coarse sand in washed and screened gravel affects the strength of concrete. Three series of tests were undertaken. 1. Effect of variation in sizing of aggregates upon the strength; 2. Effect of various amounts of grits in sand upon the strength of mortars; 3. Effect of various amounts of coarse sand in gravel upon the strength of concrete. American Concrete Institute, 1921. June, 1920–September, 1920. Cooperating agency, Indiana Sand and Gravel Producers Association.

61.232 Tests of Slag

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Study of blast furnace slags for use in concrete.

To determine the concrete-making properties of blast furnace slags from a large number of producing plants. Date of beginning, 1919. Cooperating agency, National Slag Association.

61.3 Material Survey

CALIFORNIA. *Leland Stanford University.* Mortar-making values of California sands.

Study of California sands with reference to strength of mortar and concrete. 1917–1919.

NORTH CAROLINA. *State Highway Commission.* Material survey.

Material Survey Organization: Material Survey started September, 1921, and includes location of materials suitable for all types of highway construction. It is divided under two heads: 1. Hard surface construction. 2. Sand-clay, top-soil, and gravel construction. Organization for hard surface construction: one man is sent to locate available sources within economic hauling distance of any given project as soon as the survey for location of road is authorized. He obtains all preliminary data regarding each deposit and takes samples from each deposit and reports information to sampling party which follows and takes samples from each deposit and checks the preliminary survey. Samples are sent to laboratory and tested. Information regarding the samples representing material which is acceptable for the work is given the contractor at the time the proposal is taken out. This information is in pamphlet form and consists of a map upon which is shown the line of the road and the relative location of the material deposits available for that particular project with the names of property owners and estimated quantities.

The local deposits are not guaranteed with reference either to quality or quantity, but the contractor is given all the information available and required to investigate each deposit and satisfy himself regarding its suitability. As a result of furnishing information regarding local information, savings up to sixty thousand dollars on single projects have been accomplished.

For reference purposes, all desposits are filed with the county as a unit. A list of the producers on a commercial scale is also furnished with the material survey data.

On sand-clay, top-soil, and gravel road work, the survey consists mainly of locating suitable pits to eliminate as much haul as possible and determine the most suitable material available for the given road.

OREGON. *State Agricultural College.* Concrete sands of Oregon.

PENNSYLVANIA. *State Highway Commission.* Road material survey.

The information is secured for the contractors. Similar to plan described in report from North Carolina State Highway Commission. The following forms indicate the character of the work; and information supplied.

MATERIAL INVESTIGATION

Route **Appl.** **County** **Township** **Borough**

Type

The following data concerning local materials have been compiled by the Pennsylvania State Highway Department, based on investigation and inspection of the locations and tests of the materials by Engineers of the Department. The pits and ledges have not been developed and neither the quantity nor the continuance of the quality is guaranteed.

Materials equal in quality to those designated, and meeting specification requirements from any source, will be accepted.

Acceptable materials must be furnished from other sources whenever the sources here indicated fail in quality or quantity.

TABLE OF MATERIALS

Location	Kind	Source	*Acceptable for Items No.	Tests No.	Remarks

PENNSYLVANIA STATE HIGHWAY DEPARTMENT

......................, Pa., ...19.....

Engineer of Tests:

My investigation of a deposit of..................for use on Route........
(Stone-Sand-Gravel)

Sec.........Appl.........Auth.........County................has been completed and the following data obtained: Owner's Name....................

Address.......................Location of Supply.........................
(Borough–Tw'p–County.)

Dead Haul is............Mi. to Sta...........over..........................
(Improved-dirt-form road, etc.)

with an................grade not exceeding........... Supply was sampled
(Uphill–Downhill) (Per cent)

on....................and sample shipped................. Percentage of
(Date) (Date)

deposit represented by sample is.................. Workable area of deposit is........with average thickness of................feet. Stripping would be

............ft. of..................... Strata vary from............inches
(Sand–Clay–Shale, etc.)

to..........inches in thickness and are....................... From inspection
(Horizontal -Inclined, etc.)

* Item Nos. here referred to are those given in the proposal sheet.

give extent of variation in quality of strata..................................

....

If deposit consists of boulders or field stone give (1) Stone Sizes.............

(2) Uniformity................(3) Approximate quantity....................
Structure–Kind

(4) How found........................ Water supply is..............Mi.
Walls–Piles–or Scattered

from deposit. Distance to nearest R. R. Shipping point is..............Mi.

Is material available?....................If not, state reasons why..........

....

Show by sketch of plan and section, or otherwise, (1) the part of deposit where sample was taken, (2) location and depth of test pits, (3) how stripping increases or decreases, (4) location of nearby buildings or other obstructions, (5) any other important details. (Use additional sheet if necessary.)

If material requires (1) chuting down hillside, (2) mining, (3) construction of incline, (4) new roadway, (5) improving old roadway, (6) or industrial railroad, give all details. (Use additional sheet if necessary.)

Approved.................................. Signed........................

Title..................................... Title.........................

PENNSYLVANIA. *Geological Survey.* Location of limestone suitable for road building.

To locate limestone suitable for the making of concrete in the building of State and County highways, to indicate the amount and value of the stone at various points, and the extent and cost of stripping and quarrying. Usual field equipment of geologists. Six men in the field devoting their time to study of the outcrop of Vanport limestone in the western part of the state, determining the location of possible quarry sites, studying quarries, determining the amount of stone, thickness, amount of stripping, and getting other desirable information. July, 1919–November, 1919. Cooperating agency, Pennsylvania State Highway Department.

SOUTH DAKOTA. *State School of Mines.* Tests on South Dakota sands to determine their fitness as fine aggregate in concrete construction.

A survey of sands of the State to locate desirable materials for concrete construction. To date thirty-two samples have been received. Sand sieves, briquettes and molds, cement briquette testing machine, specific gravity flasks, 1/10 cubic feet measuring box, washing flask. Colorimetric method for determining the presence of organic matter. Date of beginning, June, 1921.

WEST VIRGINIA. *State Highway Department.* Hardness tests on sand.

Comparative tests on various sands used in West Virginia. Dorry Hardness machine. Core mixture, one part cement to two parts sand. Cured in water for twenty-eight days, dried to constant weight before testing. Standard core made up of Ottawa sand in same proportions. Date of beginning, April, 1921.

GENERAL ROAD MATERIAL SURVEYS HAVE BEEN REPORTED FROM THE FOLLOWING:

University of Alabama.
Leland Stanford University.
Colorado Agricultural College.
Colorado State Highway Department.
Colorado Geological Survey.
Georgia Geological Survey.
University of Idaho.
Illinois Geological Survey.
Purdue University.
Iowa Geological Survey.
University of Kansas.
Kansas State Agricultural College.
Kentucky Geological Survey.
Kentucky Department of State Roads and Highways.
University of Michigan.
Minnesota Geological Survey.
Mississippi Geological Survey.
University of Missouri.
Missouri Geological Survey.
Missouri State Highway Department.
Montana School of Mines.
Nevada Department of Highways.
New Hampshire State Highway Department.
North Dakota State Highway Commission.
University of South Carolina, Department of Geology.
University of Texas, Department of Geology.
Wisconsin State Highway Commission.
Wyoming State Highway Commission.

61.4 Low Grade Aggregates

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Wear tests of concrete composed of low grade aggregates.

Oyster shells, coquina, cinders, burnt shale aggregate.

62.12 Tests of Tar

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* The consistency of tars as determined by the softening point and float tests.

To determine relations existing between the results of the softening point and float tests. The investigation aims for a comparison of test results, of the value of each test in distinguishing between slightly varying consistencies and accuracy of check tests. Date of beginning, 1920.

MICHIGAN. *University of Michigan.* Comparisons of the viscosities obtained by the use of Saybolt-Furol and Engler viscosimeters.

See 1922 Proceedings of the Eighth Annual Conference of Highway Engineering at the University of Michigan. On file at the Davis Library of Highway Engineering and Highway Transport, University of Michigan.

MICHIGAN. *University of Michigan.* Inter-relationship of the physical properties of tar.

See above reference.

PENNSYLVANIA. *State Highway Department.* Determination of the relative effect of petroleum and coal tar naphtha when used as a cut-back flux on asphalt.

Apparatus and reagents for asphalt and naphtha analyses, Fulweiler Binding Power Machine. Completed 1921.

PENNSYLVANIA. *State Highway Department.* To determine relative drying time of various bituminous cold surface treatment materials.

Apparatus for analyses of bitumens and naphthas. Oven for maintaining constant temperatures at 115° F., float apparatus for consistency determinations, evaporating pans, drying tests, liquid material uniformly spread in a large flat pan in an amount equivalent to 0.1 gallons per square yard, placed in an oven maintained at 115° F. Sufficient material for float determinations scraped from pans at various periods up to twenty (20) days. Completed 1922.

62.21 Production of Asphalt

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* The Refining of petroleum for the production of road oils and asphalts.

To study the refining of asphaltic and semi-asphaltic petroleums with regard to the characteristics of the residuals suitable for use in road work, the influence of variations in procedure on the nature of the products and their adaptability to various uses, and the significance of tests as indicating the quality of bituminous materials. Date of beginning, April, 1919.

62.211 Sources of Supply of Asphalts and Oils

KENTUCKY. *Geological Survey.* Rock asphalt deposits in Kentucky.

Date of beginning, Summer 1924.

62.22 Tests of Asphalts and Oils

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Exposure tests on petroleum products.

To study the changes taking place in asphalt and road oil when in service upon the road, with a view to selecting ultimately desirable characteristics such material should possess. Date of beginning, June 1, 1920.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* An investigation of the effect of light on asphalts and road oils.

To study the possibility of using effect of light as a measure of accelerated weathering action. The special apparatus for these investigations is a quarts-mercury vapor lamp made by the R. U. V. Co. This lamp emits a large percentage of its energy as ultra-violet radiation, which is allowed to strike samples placed under a sheet iron hood with the lamp. After exposure tests will be made on the samples, including determination of loss, consistency, bitumen insoluble in naphtha, fixed carbon, and such other tests as are found to be desirable. The results will be compared with data secured in other exposure tests in which the same original materials are used. Correlation of these data with those from regular tests and on the process of refining should yield valuable information on the nature of the materials.

ILLINOIS. *Isaac Van Trump Laboratory, Chicago.* Kentucky rock asphalt.

KENTUCKY. *Department of State Roads and Highways.* Rock asphalt as surfacing material.

62.221 Sampling for Asphalt and Oils

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Tests for bituminous emulsions.

The development of suitable requirements and limits for use in specifications for asphaltic emulsions. Typical samples of commercial products are examined according to present standard methods and others to be tried out, developing a procedure and suitable requirements. 1. Stability of emulsions; 2. Amount of water; 3. Consistency; 4. Properties of residue presumably formed in actual use. Date of beginning, May, 1920. Work started but now temporarily discontinued.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Comparison of volatilization tests on fluid petroleum products.

To secure comparative data on the percentage loss of 163° C. and consistency of residue by the method described in Department Bulletin 691 and by the A. S. T. M. Standard Method. Comparative tests by each of the two methods on typical products using a Freas electric oven with the revolving shelf. Ordinary tests of consistency are made on the residues. Date of beginning, June 20, 1920.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* A study of fixed carbon test.

To investigate the effect of important variables in the test with a view of standardization of the method essential to securing reliable and accurate determinations. Also to evolve a method, if possible, of obviating the foaming of certain samples. Date of beginning, July 15, 1920.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Variations in results of tests due to water contamination.

To determine the effect of water contaminating samples of bituminous materials upon the results of tests, with special reference to consistency tests.

ILLINOIS. *Department of Public Works and Buildings.* Investigation of methods for testing adhesiveness of bituminous materials and mixtures.

Date of beginning, October 2, 1921. Cooperating agency, Committee on Tests and Investigations, American Association of State Highway Officials.

KENTUCKY. *University of Kentucky.* Revision of test on road oils.

KENTUCKY. *University of Kentucky.* Methods of testing rock asphalt.

63 NON-BITUMINOUS

TEXAS. *Agricultural and Mechanical College.* The physical testing of non-bituminous road materials.

See Texas Engineering Experiment Station Bulletin No. 17.

63.1 Portland Cement

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Tests of "Super Cement." Strength tests of concrete and mortar.

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Tests of "Ciment Fondu" (French fused cement).

Strength tests of concrete and mortar; effect of alkali.

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Strength of concrete made from Slag-Portland cement.

Ground mixtures of granulated slag and cement; strength tests of concrete; effect of alkali on concrete.

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Tests of "Soliditit" (Italian cement and method of concrete road construction).

Strength tests and comparison with American cements and aggregates.

63.11 Production of Portland Cement

COLORADO. *Geological Survey.* Cement possibilities, satisfactory for the establishment of plants.

To locate cement materials for private industry and for prospective establishment of state cement plants. Chemical analysis, Kilus testing apparatus. Date of beginning, 1902.

IDAHO. *University of Idaho.* Clay and cement investigation.

See Bulletin No. 2, Bureau of Mines and Geology, *Preliminary Report on Clays of Idaho.*

63.112 Tests of Portland Cement

ILLINOIS. *Lewis Institute, Chicago. Structural Materials Research Laboratory.* Study of compression test for cement.

PENNSYLVANIA. *State Highway Department.* Effect of "hot cement" upon the setting time of concrete.

Completed 1920.

WASHINGTON. *City of Seattle, Engineering Department Laboratory.* Compression and tension tests to determine effect on strength of fine grinding of cement.

January 21, 1918–January 21, 1920.

63.1122 Preparation and Treatment for Tests of Portland Cement

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Study of proposed compression test for cement.

To study the proposed compression test for cement as a possible substitute for the standard tension test as now made.

ILLINOIS. *Lewis Institute, Chicago, Structural Materials Research Laboratory.* Tests of bin-stored cement.

Frequent samples from 2,000-bbl. bin at Lehigh Portland Cement Company's mill at Ormrod, Pa.

IOWA. *Highway Commission.* Possible new early tests for cement.

To determine a test for cement which will give consistent results in a shorter period of time than that required by standard tests now in use. Date of beginning, 1920.

KENTUCKY. *University of Kentucky.* Accelerated test on cement and sand test.

WEST VIRGINIA. *State Highway Department.* Compression test on cement mortars.

Comparative tests of tensile and compressive strengths. Regular 2"x4" cylindrical molds. Procedure according to American Society for Testing Materials. Date of beginning, April, 1922.

63.1123 Methods of Tests of Portland Cement

IOWA. *State College.* Comparative cement tests.

Comparison with similar work done at other stations. Date of beginning, 1915.

63.1125 Fundamental Properties of Portland Cement Concrete

ILLINOIS. *Lewis Institute, Chicago, Structural Materials Research Laboratory.* Concrete making properties of twelve different brands of Portland cement.

Variation in strength of concrete from different brands; variation in strength of concrete from different samples of same brand.

63.211 Production of Brick

IOWA. *State College.* Paving brick manufacture.

To show, if possible, that satisfactory brick can be made from Iowa shales in Iowa plants at cost competing with Eastern brick. Complete clay-working plant includes newly built kiln. Clays and shales from various Iowa plants are worked in the laboratory to develop a method which produces satisfactory paving brick economically. Date of beginning, 1920. Cooperating agency, clay manufacturers of Iowa.

63.2123 Methods of Tests of Brick

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Investigation of the proposed impact test for paving brick.

To develop an impact or toughness test for vitrified paving brick with the idea of substituting it eventually for the standard rattler test.

NEW YORK. *Columbia University.* Tests on brick. (Material tests.)

63.3 Wood Block

MINNESOTA. *Minneapolis Department of Highways and Streets* Use of cut back pitch filler for creosoted wood block paving.

See Proceedings of American Society for Municipal Improvements.

MINNESOTA. *Minneapolis Department of Highways and Streets.* Use of 2″ creosoted wood block on bituminous cushion.

Will lay 100 yards on a heavy automobile traffic street, Harmon Place in Minneapolis.

63.4 Steel

INDIANA. *State Highway Commission.* Use of the Brinnel Meter in shop and field inspection of steel.

63.42 Tests

INDIANA. *Purdue University.* Investigation of rail bar reinforcing steel.

Bending and tensile tests of reinforcing bars rolled from steel rails. Several mills sampled from the rolls. Study of strength uniformity, as affected by location of bar in rail at progressive passes, formed bar, form of bar, size of rail, etc. See Proc. Am. Soc. Testing Materials, 1913.

63.712 Tests of Paint

IOWA. *Highway Commission.* Service tests of bridge paint.

To determine serviceability of various paints under actual weathering conditions. Usual field weathering tests, using a standard size of sheet metal fastened in racks. Date of beginning, 1914.

WEST VIRGINIA. *State Highway Department.* Protective coatings for highway bridges.

To determine the paints most suitable for use on steel bridges in West Virginia. Steel panels 10"x20"x1/8". Panels cleaned of all rust and mill scale by sand blast. Three coats of paint used. Each application of paint allowed to dry for one week before next coat was applied. All work done indoors. Panels exposed to weather by suspending a large wood frame. Paints used: samples made by various manufacturers to conform with State Bridge Standard Specifications and other special paints. Date of beginning, June 1, 1922.

WEST VIRGINIA. *University of West Virginia.* Investigation of the paints for steel highway bridges.

To determine the wearing qualities of various paints. Date of beginning, 1922.

64.1 Concrete

64.11 Bituminous, Theory of Proportioning

CALIFORNIA. *San Francisco Department of Highways and Streets.* Asphaltic concrete mixes.

To determine the most suitable proportions for the varying conditions of climate and traffic found here. Standard laboratory equipment for testing asphalts and aggregates. Date of beginning, 1912.

DISTRICT OF COLUMBIA. *Commissioners of the District of Columbia.* Design of asphaltic mixtures.

To furnish a method of determining the correct amount of bitumen and filler to use with any aggregate, thereby making any unnecessary grading in accordance with arbitrary granulometric formulae. Usual equipment for testing asphaltic mixtures and in addition a special compression machine. Comparison of new mixtures with known standards representing cracked, good, and pushed pavements. Proof of new mixtures by trial on the street. Date of beginning, January, 1920.

NEBRASKA. *Omaha Testing Laboratories.* Ideal and theoretical gradings of aggregates for bituminous pavements.

To determine the best and densest grading of materials for use in bituminous pavements. Nine sizes of materials ranging from dust to the 1½ inch size, mixed in several proportions. Some 1000 trials were made. Jan. 1, 1922–July 1, 1922.

NEBRASKA. *Omaha Testing Laboratories.* A curve of maximum density.

This work is being done in connection with "Ideal and Theoretical Gradings of Aggregates for Bituminous Pavements."

PENNSYLVANIA. *State Highway Department.* Relation in fine graded bituminous mixes, of varying analyses and containing different asphalts to deformation, toughness, density, voids, and absorption.

Development of series of tests to determine proper design of fine graded bituminous mixtures. 20,000-lb. compression machine, Page impact machine, deformation machine for applying static load, electrically heated moulds, ovens, balances, etc. Two bituminous sands of similar mechanical analyses and different composition and shape of grains mixed with five different asphalts of practically the same consistency. Mixtures were made with constant asphalt content and varying filler from 8%–15% and vice versa with each type of asphalt. Tests made for density, voids, toughness and deformation and under a falling and static load. Impact (toughness) tests at different temperatures and absorption. Completed 1922. Cooperating agency, Bituminous Sub-committee, American Association of State Highway Officials.

TEXAS. *Agricultural and Mechanical College.* Bituminous paving mixtures

The preparation and dissemination of information regarding the theory and practice in designing the popular hot mixed asphaltic types of pavements, including control of mixing processes. See Bulletin No. 28 of above college. Begun April 15, 1922.

64.12 Method of Tests

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Tests of compressed bituminous mixtures.

To investigate laboratory tests on compressed bituminous mixtures, and develop such tests as will give a measure of the properties of bituminous mixtures when under traffic. Date of beginning, 1920.

64.15 Admixtures

ILLINOIS. *Isaac Van Trump Laboratories, Chicago.* Use of pulverized copper sulphate in asphalt paving mixtures.

MICHIGAN. *University of Michigan, State Highway Laboratory.* Effect of copper sulphate on asphalt cement.

Completed July 1, 1922.

64.2 Portland Cement Concrete

ILLINOIS. *University of Illinois.* General Investigation of Concrete.

See Bulletins of the Engineering Experiment Station of the University of Illinois for important fundamental studies, by Prof. A. N. Talbot, of the principles governing design and use of concrete, plain and reinforced.

ILLINOIS. *Lewis Institute, Chicago.*

See Bulletins of the Lewis Institute and technical papers by Prof. Duff Abrams, Dr. J. C. Witt, and Mr. Stanton Walker for important studies and investigations of cement and concrete.

OHIO. *State Highway Department.* Investigations on concrete, slag, etc.

64.21 Theory of Proportioning

DISTRICT OF COLUMBIA. *Commissioners of the District of Columbia.* Proportioning concrete for bases and one-course pavement.

Object is economy and strength. Date of beginning, January, 1922. Cooperating agency, U. S. Bureau of Public Roads.

ILLINOIS. *Department of Public Works and Buildings.* Investigations in gradation of concrete aggregates to determine the effect on strength of concrete.

700 specimens tested in compression and flexure. January 20, 1922–May 4, 1922.

IOWA. *State College.* Proportioning pit-run gravel concrete.

See Bulletin No. 60, Iowa Engineering Experiment Station.

IOWA. *State Highway Commission.* Use of different ratios of sand to coarse aggregate in concrete pavements.

To determine for concrete mixes the effect of varying ratios of sand to coarse aggregate upon the strength and resistance to wear of the concrete. Beams, cylinders, and wear blocks; consistency by cone and flow-table. Date of beginning, December, 1921.

IOWA. *State Highway Commission.* Methods of proportioning concrete materials.

See Bulletin No. 60, May, 1921, Engineering Experiment Station, Ames, Iowa. See above, Iowa State College.

IOWA. *State Highway Commission.* Investigation of mortar-void theory of proportioning.

To find method of using poorly graded sands in concrete. September, 1921, date of beginning.

IOWA. *State Highway Commission.* Theory for estimating amounts of material in concrete.

See Service Bulletin, Iowa Highway Commission, March, 1921.

IOWA. *State Highway Commission.* Practical use of excess sand in concrete mixtures.

See Engineering News-Record, November 17, 1921 and Proceedings of American Society for Testing Materials, 1922.

KENTUCKY. *University of Kentucky.* Check on water content and slump test as proposed by Abrams for proportioning concrete.

MAINE. *University of Maine.* Classification of Maine sands as regards mineral content, surface-area, and fineness-modulus.

To determine the relative merits of surface area and fineness-modulus on Maine sands of given mineral constituents. Date of beginning, November, 1922. Cooperating agency, Maine State Highway Commission.

MINNESOTA. *University of Minnesota.* Study of concrete mixtures, by method of trial mixes.

To determine the practical method for designing mixtures in the field and to compare the strength obtained by these mixtures with that of other methods now being proposed. Date of beginning, 1918.

NEBRASKA. *Department of Public Works.* Platte river sand-gravel aggregate, as an aggregate for concrete.

To show strength variation as a function of sizes of aggregate.

NEBRASKA. *Omaha Testing Laboratories.* Experiments on Nebraska pit-run gravels.

To design concrete from sand-gravel. To determine the best possible grading should give the strongest concrete. 500 experiments covering all possible physical tests made on sand, gravel, and a mixture of the two. January 1, 1921–May 1, 1922.

OHIO. *State Highway Commission.* Variation in proportions, consistency, etc., of concrete.

SOUTH DAKOTA. *State College.* Methods of proportioning concrete.

To develop principles and methods of proportioning which may be more successful from the standpoint of practical application. Methods not essentially new, but field limitations of laboratory aimed to be overcome. Date of beginning, September, 1921.

TEXAS. *University of Texas.* Principles of pavement selection with statistics of pavements in Texas cities and towns prior to January 1, 1920.

Bulletin 23, Texas Engineering Experiment Station, May 1, 1920.

TEXAS. *University of Texas.* Basic principles of concrete design.

Shown in Bulletin No. 1815. To determine where the inflection point occurs, what the strength of the concrete is for that degree of richness, and what the slope of the curve is at that point. Date of beginning, 1919.

WASHINGTON. *City of Seattle Engineering Department Laboratory.* Effect of large-size aggregate on the strength of Portland cement concrete.

Gravel aggregate up to 6" size. Compression cylinders 6"x12". Beams 48"x 17"x6" to 9". Tested at 28 days. October 1921. 4" gravel is standard in City of Seattle.

WISCONSIN. *State Highway Commission.* Strength and abrasion tests on Lannon dolomite in concrete.

To determine: (1) the effect upon wear and strength in mixes used for one-course road construction by increasing the amount of coarse aggregate; (2) the comparative strengths and wearing resistance of concrete when tested at 21 days and at 28 days; (3) the effect of larger size stone (1¼" and 2" diameter) on abrasive wear. In all, a total of 48 slabs, varying by volume proportion and grading of coarse aggregate and a like number of cylinders were made, one cylinder and one slab to a batch. Proportions, 1–2–3½ up to 1–2–4¼. Completed June 12, 1920.

WISCONSIN. *State Highway Commission.* Effect of finely graded sands on strength and abrasion resistance of concrete for one-course road construction.

To determine if by adjusting suitably the proportions of cement, sand, and coarse aggregate, a concrete suitable for this type of construction can be obtained, using sands hitherto considered unwise, containing more than 20% passing a No. 50 mesh sieve and over 5% through a No. 100 mesh sieve. A comparison between wear of Lannon stone and gravel. The proper mix for materials determined. Slabs and cylinders cast of varying mix, tested at 21 days. Completed June 11, 1920.

64.22 Methods of Tests

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Determination of the most suitable abrasion test for concrete.

ILLINOIS. *Lewis Institute, Chicago, Structural Materials Research Laboratory.* Effect of size and shape of test piece on the compressive strength of concrete.

IOWA. *State College.* Chemical determination of the cement content of concrete.

To devise a method for the determination of the cement content of concrete by chemical analysis of samples of the concrete. Date of beginning, 1913.

KANSAS. *State Agricultural College.* Wear test of concrete.

To devise a standard wear test for concrete. Concrete of desired proportion is molded into 9" spherical specimens and tested in standard brick rattler. Proportions and curing treatment may be varied as desired. Date of beginning, December, 1919.

PENNSYLVANIA. *State Highway Department.* Determination of the relation of height to compressive strength on concrete cylinders.

To determine the relative value of plaster of Paris and neat cement plus calcium chloride for capping concrete cylinders having irregular bearing surfaces. Effect of Mattimore's impact test on strength of concrete prior to making compressive strength tests. Relation of height of concrete cylinders to compressive strength. 200,000 lb. compression machine, Mattimore's impact machine. A series of 6" diameter cylinders of concrete having the same composition and consistency were cast in heights of 5, 6, 7, 8, and 12. Sufficient of each set were moulded of uniform bearing and irregular bearing for all tests. Each set was tapped with plaster of Paris and with neat cement containing 4% calcium chloride. Compressive tests run and cylinders subjected to impact prior to crushing were capped with calcium chloride. Completed 1920.

TEXAS. *University of Texas.* The effect of rodding concrete.

See Proceedings of the American Society for Testing Materials, Volume XX, pages 219–232; Volume XXI, pages 1006–1012.

64.221 Field Tests and Cores

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Relation between the wear and strength of concrete in laboratory and in actual construction.

GEORGIA. *School of Technology.* Strength of concrete pavements as tested by cores drilled from same.

To determine the value of different methods of finishing and of different materials in concrete roads. Blocks, about 2 feet square, will be taken from finished roads, 4" cores will be drilled at laboratory and tested for compression and density. Date of beginning, July 1, 1922. Cooperating agency, Fulton County, Georgia.

ILLINOIS. *Department of Public Works and Building, Division of Highways.* Comparison of molded concrete cylinders with cores drilled from road.

KANSAS. *State Agricultural College.* Concrete used in highway construction.

A study of the properties of concrete used in highway construction, to determine the effect of using various materials and methods. Test cylinders are taken during construction and cores from finished pavement after completion. These samples are studied in the laboratory and the results compared with materials used and different methods of construction. Date of beginning, 1920.

MINNESOTA. *University of Minnesota.* Investigation of one-course concrete pavement constructed by the state during the past three years.

To determine the quality and properties of concrete in one-course concrete pavements constructed by the state during the past three years. Core drilling machine, compression machine, Berry strain gauge, constant temperature device. About four hundred 4½ cores were drilled out of the pavements and the following points investigated: (a) thickness of slab, (b) depth of mortar on top of slab, (c) density of concrete, (d) porosity of concrete, (e) expansion due to heat and moisture, (f) modulus of elasticity, (g) compressive strength of cores, (h) compressive strength of cylinders made from concrete on pavement while being constructed. Date of beginning, June 1921.

NEW JERSEY. *State Highway Commission.* The strength secured in the field of concrete prepared from different aggregates and of different proportions.

To determine the strength of concrete required to carry different types of travel without abnormal deterioration.

OREGON. *Portland Department of Highways and Streets.* Comparative tests of concrete pavements.

Samples cut from finished work by diamond drill.

PENNSYLVANIA. *Altoona Department of Highways and Streets.* Compressive strength tests of concrete as placed in the pavement.

Crushing of 6″x12″ concrete cylinders at 7- and 28-day periods.

PENNSYLVANIA. *State Highway Department.* To determine relative strength of cores taken from the road after acceptance and cylinders made when constructed.

Portable core drill, cylinder molds, testing machine. Date of beginning, 1919.

WISCONSIN. *State Highway Commission.* Tests on concrete cylinders made at several jobs.

To secure information concerning the strength and uniformity of field concrete, also to secure information on proportions used, moisture in fine aggregates, etc. Instructions for making concrete cylinders and auxiliary tests were compiled. Completed April 7, 1921.

WISCONSIN. *State Highway Commission.* Wearing resistance and strength of concrete made on paving jobs and similar tests of concrete made of like materials in the laboratory.

To ascertain the wearing resistance and strength of concrete made in pavements and cured under the same conditions as the pavements (field made). Comparison of wearing resistance and strength of field-made concrete with the same properties of a laboratory-made product. To collect data concerning location of pavement from which specimens were taken, and the conditions under which they were made, for the purpose of rating the significance of the rattler test for abrasive resistance of concrete. To secure information concerning the proportions and consistency of concrete and methods of operation as used on the job. A field party was organized with a motor vehicle containing platform scales, spring balance, thermometer, consistency cone, moulds, and other apparatus. Six 4¼″x19½″ slabs and six 6″x12″ cylinders were made at each job, two each on three successive days. The slab forms were placed on the subgrade and filled with concrete as it was dumped for the pavement, and finished in regular manner. These specimens were withdrawn on the succeeding day and cured in the same manner as the pavement. Other observations were: weight per cubic foot of aggregate, per cent of moisture in fine aggregate, proportions used, consistency of concrete. Samples of materials were afterwards tested in the laboratory as well as the test specimens moulded in the field, which were duplicated by laboratory methods from materials collected in the field. Completed January 29, 1921.

64.23 Fundamental Properties

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Effect of temperature changes on concrete slabs.

Concrete slab, 18x24 feet, recording thermometers imbedded at different distances from the top of the slab. Changes in elevation are recorded by solid brass plugs at different points with a precise level. Movement is measured by recording devices and Ames dials.

ILLINOIS. *Department of Public Works and Buildings, Division of Highways.* Thermo-conductivity of concrete.

To determine at what temperature concrete containing various percentages of calcium chloride will freeze.

ILLINOIS. *Department of Public Works and Buildings, Division of Highways.* Fatigue of concrete.

To determine if concrete beams will rupture under a large number of applications of loads of less magnitude than necessary to cause failure when applied statically. Results apply to working stresses for concrete roads under truck traffic. Loads of various percentages of static breaking load applied to ends of seven plain concrete cantilever beam specimens 6″x6″ cross-section and 36″ long, mix 1-2-3½. Loading apparatus consists of a pair of automobile wheels carried on a horizontal axle and revolved about a vertical shaft, concentric with supported ends of beams. Blocks of concrete flush with the beams, provide a smooth track for the wheels which load each specimen forty times a minute. An autographic record of deflections is obtained. Date of beginning, June 15, 1921.

ILLINOIS. *Lewis Institute, Chicago, Structural Materials Research Laboratory.* Yield and density of concrete. Concrete made from different aggregates.

ILLINOIS. *Lewis Institute, Chicago, Structural Materials Research Laboratory.* Transverse strength of concrete.

Report published in Proceedings American Concrete Institute, 1922, and Bulletin 11 of Structural Materials Research Laboratory.

INDIANA. *Purdue University.* Ball tests of mortar and concrete to determine surface strength.

To determine the surface strength of mortar and road concrete before and after the material has been subjected to a truck wheel load. The ball test is made by noting the load required to push a ½″ steel ball into the mortar or concrete to a depth of ¼″. A study is being made by the use of this test to the surface strength of a concrete before and after a truck wheel load has been placed on the concrete. Date of beginning, June 1921. Cooperating agency, U. S. Bureau of Public Roads.

INDIANA. *Purdue University.* Fatigue tests of mortar and concrete beams.

To study plain mortar and concrete in a fatigue test. A specially designed machine to apply alternating stresses in tension and compression to a concrete beam. A concrete beam is subjected to bending first in one direction, then in the reverse direction, under a load equal to a definite percent of the static breaking load. Date of beginning, June 1921. Cooperating agency, U. S. Bureau of Public Roads.

MAINE. *University of Maine.* A study of the variability of different mixtures of sand and cement in compression.

A study made on the variability of different 2" x 4" compression cylinders in the following mixes: 1:3, 1:2½, 1:1½, 1:1. Date of beginning, February, 1922. Study nearly completed. Cooperating agency, Maine Agricultural Experiment Station.

MARYLAND. *University of Maryland.* Fatigue of concrete.

To ascertain the effect on elastic properties of Portland Cement mortar due to repeated strain, such as occurs in concrete slabs on highways due to traffic loads. Specimens are 3" x 4" x 24", mix 1–2. Special apparatus designed and built in the laboratories of the University of Maryland. Two beams are shackled together at both ends and are deflected small amounts, from 0.002 of an inch to 0.01 of an inch by an eccentric shaft placed between the centers of the length. At intervals during the test the beams are put in a static machine to measure change in elastic properties. The elastic range of stress is determined by delicate mirror extensometers, to study its relation to the fatigue limit. Date of beginning, Summer 1921. Cooperating agencies, State Road Commission of Maryland and U. S. Bureau of Public Roads.

MASSACHUSETTS. *Sherman, Skinner, Esselen, Inc., Boston.* Increasing elasticity of concrete roads.

To obtain an easier wearing surface without decrease of strength. Procedure is secret. Date of beginning, 1922.

MINNESOTA. *University of Minnesota.* Shrinkage and time effect on concrete.

To determine the shrinkage and plastic flow of concrete under sustained load and also to determine, if possible, how shrinkage can be eliminated. Concrete slabs and strain gauges. Testing machines. A study of shrinkage in plain concrete over periods up to eight years. These tests cover a study of the effect of variations in (a) proportions, using neat cement mortars and concretes of usual mixes; (b) curing, such as dry air with variations in temperatures, moist air or water, and steam curing; (c) kinds of coarse aggregates; (d) shape of specimens and direction and position of gage line measurements; (e) percent of water used; (f) density of concrete; (g) surface tension of the water used; (h) foreign ingredients such as hydrated lime, plaster of Paris, etc. Date of beginning, 1914.

MINNESOTA. *University of Minnesota.* Study of shrinkage and plastic time effect in reinforced concrete columns.

A study is being made to clear up points in design which are now under consideration by the joint committees of the American Society of Civil Engineers and the American Society for Testing Materials. Several columns have been designed and built in the laboratory in order to obtain data regarding the manufacture of concrete and the curing treatment. These columns are to be tested under continuous load. The work is being carried on by observing the actions of the columns under stress with Ames dial strain gages. Date of beginning, 1914.

PENNSYLVANIA. *Lafayette College.* Effect of vibrations during time of setting and strength of concrete.

To see if the vibrations caused by passing trains have any objectionable effect upon concrete which is poured upon the bridge and left to set while the bridge is undergoing vibrations caused by passing trains. Test pieces will be poured on bridges set apart for that purpose and other tests will be made and kept under constant vibration by means of a pneumatic hammer. January 1, 1922, to August 1, 1923. Cooperating agency, Pennsylvania Railroad.

PENNSYLVANIA. *Lehigh University.* Pull-out tests of straight and hooked bars imbedded in concrete specimens.

PENNSYLVANIA. *State Highway Department.* Absorption test of concrete.

To determine the relative absorption of concrete made of various coarse aggrevates. 1921–1922.

TEXAS. *University of Texas.* The flow of heat into and in concrete.

Experiments with resistance wires as a means of determining temperature in concrete and other materials. See *Heating and Ventilating Magazine*, August, 1922. A concrete slab 10 x 16 feet and 9 inches thick has been poured, to represent a section of concrete highway, with eight sets of resistance wires to measure temperatures and with five rows of metal plugs to measure deflections. Temperature readings were taken during the setting of the concrete and readings will be taken occasionally during 24-hour periods to determne variations in temperature and form. An elevated slab to represent a concrete bridge will be constructed later for similar studies.

TEXAS. *University of Texas.* The strength of concrete as determined by the strength of its coarse aggregates.

For preliminary results see University of Texas Bulletin No. 2215, or *Engineering News-Record*, June 29, 1922. The three months tests of a series of aggregates ranging from very soft limestone to trap have just been completed.

TEXAS. *University of Texas.* Physical properties of dense concrete.

See University of Texas Bulletin No. 1815.

TEXAS. *University of Texas.* The strength of fine aggregate concrete.

See University of Texas Bulletin No. 1855.

WASHINGTON. *City of Seattle, Engineering Department Laboratory.* Transverse and compression tests on concrete to determine effect of maximum size aggregate.

To determine effect of size of aggregate on compression, tension, and shearing strength and also wear in concrete for pavements. Small Wonder mixing machine, 300,000 lbs., Rieble testing machine. Slabs for transverse test 17″ x 48″ with varying thickness; 6″ x 12″ concrete cylinders. December 8, 1919–February 21, 1920.

WISCONSIN. *State Highway Commission.* Effect of age on wearing resistance of concrete slabs, previously tested when 28-days old.

Due to the accumulation in the laboratory of a large number of slab specimens, left from various tests, it was thought worth while to determine the effect of age on the wearing resistance of these specimens in order to have some criteria of the amount of loss which such specimens would sustain if tested at ages older than 28 days. Also to ascertain if concrete made from certain aggregates might wear proportionately more after aging than concrete made from other materials. Completed June 30, 1921.

64.238 Wear Tests on Concrete

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Wear tests on pavement sections.

To determine resistance of brick, granite block, and concrete to steel tire traffic. Comparison of results of laboratory tests with behaviour as pavements. Special abrasion machine consisting of five cast-iron wheels, each weighing 1,000 lbs., 4′ diameter and 2″ thick. In a trackway 2′ wide by 400′ long, forty-eight sections of pavement were laid: twenty-one brick, nineteen block, and eight concrete. These were subjected to action of the abrasion machine which was drawn over them at a rate of five miles per hour. To be continued with additional sections of material. March, 1919–July, 1920.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Wear tests of concrete.

A series of tests in the Talbot-Jones rattler on various mixes of concrete to determine resistance to wear.

INDIANA. *Purdue University.* Wear and compression tests of concrete.

To determine the wearing quality of a 1-2-3 concrete, using differently graded gravel aggregates when the concrete was formed in wedge-shaped blocks. A modified Talbot-Jones rattler. Ten wedge-shaped blocks having an 8″ x 8″ wearing surface, circular bottom, and a depth of about 5¾ inches were arranged in a perimeter of a steel band, which in turn was placed in a revolving drum. 200 lbs. of steel shot to conform to the American Society for Testing Materials requirements for the rattler test of paving brick was used in the abrasive charge. The depth of wear was computed from the loss in weight of the blocks after revolving 1800 revolutions. June, 1919–September, 1919.

MISSOURI. *University of Missouri.* Wearing qualities of concrete.

To determine wearing qualities of concrete made from different kinds of aggregates from various sections of the State. Spherical molds, 9″ diameter, cylinder forms, 6″ x 12″, brick rattler, compression testing machine, Deval abrasion machine, etc. Abrasion tests in brick rattler of 9″ spheres of concrete to determine percent of loss; and comparison with Deval abrasion tests of stone used in making concrete to determine relation between quality of concrete and aggregates, if any. Date of beginning, January, 1922.

IOWA. *State College.* Resistance of concrete to wear.

To determine wearing-properties of Iowa limestones for concrete road surfaces. Talbot-Jones rattler. Regular wear test. Date of beginning, 1920. Cooperating agency, Iowa State Highway Commission.

PENNSYLVANIA. *State Highway Department.* Impact tests upon concrete cylinders.

To determine the resistance of concrete to wear by impact tests made on moulded cylinders and cores drilled from the road. Mattimore impact machine, portable core drill, testing machine. Date of beginning, 1919. Proceedings A. S. T. M., 1920, p. 266.

TEXAS. *University of Texas.* Studies of wear-resisting properties of concrete aggregates.

Coarse aggregates: to determine the relation between the physical properties of the rock and maximum size of the aggregate as well as the thickness of mix, on wear and strength of concrete. Fine aggregate: effect of kind of fine aggregate and grading of sand on wear and strength of concrete. Relation between mortar tests of sand and wear resistance of concrete. Physical apparatus and Talbot-Jones rattler. Four wear test slabs and 6 x 12-inch compression specimens will be made from each combination of fine and coarse aggregate, tested at the ages of 28 days and of 3 months. Thirteen coarse aggregates, ranging from 1.0%–9.5% wear, as well as four kinds of Texas sands will be used. January 1, 1922–November 30, 1922.

WASHINGTON. *State College.* Study of abrasion tests on concrete spheres, using brick rattler.

Cooperating agency, State Highway Department.

64.239 Consistency

DISTRICT OF COLUMBIA. *U. S. Bureau of Standards.* Consistency of concrete.

Study of the factors which enter into consistency for purpose of securing better control. Flow table. Begun spring of 1919.

ILLINOIS. *University of Illinois.* Measurement of mobility of fresh concrete.

INDIANA. *Purdue University.* Comparison of flow table and cone slump for measurement of consistency of concrete.

MAINE. *University of Maine.* Development of a field and a laboratory method for determining the workability of concrete.

To determine the lubricating effect of water, cement, and aggregate to produce a workable mixture of concrete. Truncated cone with adjustable drop and a slip plane adjustable to 0 degrees to 50 degrees angle. Drop sample, measure diameter and depth and slide down plane, noting angle started, angle of sliding, character of pat as it moves along incline, whether sliding, slipping, or rolling. Date of beginning, March, 1922. Cooperating agency, Maine State Highway Commission.

WISCONSIN. *State Highway Commission.* Effect of consistency on the abrasive resistance and strength of 1-2-3½ concrete.

Coarse aggregate, gravel, crushed dolomite, and crushed granite. Fine aggregate, well-graded sand, and fine sand of poor grading. Four consistencies, ¾" slump, and the water increased by 5%, 10%, and 20%. Consistency measured by cylinder slump, cone slump, flow table, and angle of repose of 6" x 12" freshly moulded cylinder on galvanized iron plane. Completed December 21, 1920.

64.24 Destructive Agencies

COLORADO. *Agricultural College.* Effect of beet pulp upon concrete.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Effect of alkali water on concrete.

The alkali water is used for mixing and also as storage water to determine its effect on mortar specimens.

DISTRICT OF COLUMBIA. *U. S. Bureau of Standards.* Durability of cement drain tile and concrete in alkali soils.

Both drain tile and concrete blocks were placed at different locations in the United States and Canada, being subjected to action of alkali as well as fresh water. Date of beginning, 1913. Cooperating agencies: U. S. Bureau Public Roads, U. S. Reclamation Service, Engineering Institute of Canada, Portland Cement Association, American Concrete Pipe Association.

ILLINOIS. *Lewis Institute, Chicago, Structural Materials Research Laboratory.* Use of impure waters for mixing concrete.

ILLINOIS. *Lewis Institute, Chicago, Structural Materials Research Laboratory.* Tests of concrete exposed to alkali.

Experimental studies of concrete exposed to alkali soils and waters; field tests in Colorado, South Dakota and Western Canada. Specimens stored outdoors in tanks at Laboratory. Effect of different mixtures, consistencies, curing conditions, aggregate, etc. Effect of integral alkali and waterproofing compounds on strength of concrete. Theoretical studies, largely chemical. Chemical analysis of materials used in concrete tests.

IOWA. *State Highway Commission.* Effect of quality of mixing water upon concrete.

Effect upon the strength of concrete of various chemicals found in the natural waters of the state. Date of beginning, December, 1921.

64.246 Climatic

WASHINGTON. *City of Seattle, Engineering Department Laboratory.* Effect of running water on concrete.

To determine the effect of local water upon concrete and the life of concrete used in drain pipes and bulkhead construction subjected to running water continually. 2″ x 4″ mortar cylinders tested under compression and standard briquette tests. November 14, 1921–November 14, 1926.

WISCONSIN. *State Highway Commission.* Rattler tests on slabs made of various kinds of aggregates before and after exposure to weather.

To secure information concerning the relative durabilities of the various concretes tested. To ascertain the effect of alternate freezing and thawing and wetting and drying on the worn surface of concrete road slabs made of various types of aggregates when the slabs were exposed with the worn faces upward on a former marsh. They were then retested in the rattler with 150 lbs. of small shot and 50 lbs. of large shot and six slabs. Several coarse stone aggregates; gravel and mine tailings. The rattler is a modification of the Standard brick rattler in which a set of six blocks are set circumferentially and subjected to the rolling and impact of the shot. Dust is allowed to escape. The charge is given 1800 revolutions in the clockwise and 1800 revolutions in the counter-clockwise direction. Completed May 17, 1921.

64.25 Admixtures

DELAWARE. *State Highway Commission.* Use of "Cal" as an accelerator in cold weather.

Investigation completed with tests of concrete mixed at 36 degrees F. and containing different percentages of "Cal" and stored outside until tested.

DISTRICT OF COLUMBIA. *U. S. Bureau of Public Roads.* Study of the effect of treating concrete for the purpose of preventing contraction due to drying out.

To determine the effect of admixtures of certain substances such as hydrated lime and "Cal" in concrete for the purpose of preventing too rapid drying out. To ascertain the effect of surface treatment of concrete with bituminous material such as tar for the same purpose.

DISTRICT OF COLUMBIA. *U S. Bureau of Public Roads.* Study of the effect of admixtures of various materials in concrete.

To study the effect of admixtures of various substances with Portland Cement for the purpose of (1) accelerating the strength, (2) preventing premature drying out of concrete and (3) increasing workability.

ILLINOIS. *Department of Public Works and Buildings.* Testing of patented materials.

March 30, 1921–April 23, 1921.

ILLINOIS. *Lewis Institute, Chicago, Structural Materials Research Laboratory.* Tests of calcium chloride in concrete.

MASSACHUSETTS. *Skinner, Sherman, Esselen, Inc., Boston.* Waterproofing concrete roads.

To render concrete waterproof without reduction in strength. Procedure is secret. Date of beginning, 1922.

MICHIGAN. *University of Michigan, State Highway Laboratory.* Effect of calcium chloride on hardening of concrete.

Compressive strength apparatus. Completed July 1, 1922.

WASHINGTON. *City of Seattle, Engineering Department Laboratory.* Effect of "Cal" on concrete or mortar.

2" x 4" mortar cylinders tested under compression and standard briquette tests. March 6, 1922–May 6, 1922.

DISTRICT OF COLUMBIA. *National Lime Association.* Effect of hydrated lime upon strength of concrete.

DISTRICT OF COLUMBIA. *National Lime Association.* Internal stresses due to variations in the moisture content of concrete containing hydrated lime.

ILLINOIS. *Department of Public Works and Buildings.* Effect of various percentages of lime on strength and volume constancy of concrete.

Fine and coarse sands as aggregates. Compression and flexure tests. Lime percents, 2½, 5 and 7½. February 1, 1922–May 8, 1922.

ILLINOIS. *Lewis Institute, Chicago, Structural Materials Research Laboratory.* Effect of hydrated lime and other powdered admixtures in concrete.

NORTH CAROLINA. *State Highway Commission.* Hydrated lime in concrete.

See appendix of report of Committee 67, American Society for Testing Materials, 1921.

SOUTH CAROLINA. *University of South Carolina.* Effect of varying proportions of hydrated lime upon tensile strength of cement-sand mortars.

To make 750 grams of sand, 250 grams of cement (plus percentages 1–10 or 1-12 of hydrated lime) into plastic mortar of normal consistency and from this into briquettes, six at a time; to store these in accordance with standard specifications; to test these at 7, 28 and 56 days for tensile strength. Date of beginning, April, 1921. Cooperating agency, State Highway Department of South Carolina.

SOUTH DAKOTA. *State Highway Commission.* The effect of hydrated lime on mortar (setting and strength).

Strength of mortar, adhesion of old and new cement mortar, strength of cement mortar when organic matter is present, time of setting.

SOUTH DAKOTA. *State Highway Commission.* The effect of hydrated lime on time of absorption of concrete.

WASHINGTON. *City of Seattle, Engineering Department Laboratory.* Compression test to determine effect of hydrated lime in concrete.

May 3, 1919–June 3, 1919.

WISCONSIN. *State Highway Commission.* Effect of hydrated lime on mechanical properties of concrete for road construction.

To ascertain the effect of additions of small percentages of hydrated lime on the following properties of concrete, of the quality used in one-course road construction: on its workability, resistance to segregation during transportation, capacity to withstand a dry atmosphere during curing, transverse strength and resistance to wear. Consistency by flow table. Tendency to segregation measured by jigging table. Concrete proportioned 1-2-4 with two flowabilities, 160 (pavement) and 180 (rather wet). Hydrated lime added 5% and 10% of weight of cement. Aggregates, sand and crushed dolomite. Completed July 22, 1921.

PART V.

TABLE SHOWING THE NUMBER OF PROJECTS IN THE DIFFERENT FIELDS OF HIGHWAY RESEARCH.

	Economics	Operation	Design (road)	Construction	Material	Total
Colleges and universities.........	15	4	48	3	114	184
Industrial.....................	3	2	5	1	21	32
Municipalities.................	3	3	8	7	10	31
Federal and State Highway Depts	33	9	53	5	105	205
State geologists...............	0	0	0	0	17	17
Counties......................	4	0	5	1	0	10
Total.................	58	18	119	17	267	479

PART VI

TECHNICAL AND EDUCATIONAL ORGANIZATIONS AND COMMITTEES ACTIVE IN THE FIELD OF HIGHWAY RESEARCH AND TECHNICAL DEVELOPMENT

American Association of State Highway Officials
- Committees
  - (1) Executive
  - (2) Standards
    - (a) Plans and surveys
    - (b) Design
    - (c) Specifications
    - (d) Traffic control and Safety
    - (e) Bridges and Structures
  - (3) Administration
  - (4) Construction
  - (5) Maintenance
  - (6) Tests and Investigations
    - (a) Abrasion Tests for Gravel and Stone
    - (b) Shale in Fine and Coarse Aggregates
    - (c) New Tests for Cements
    - (d) Bituminous Tests
  - (7) Cooperation with Contractors
  - (8) Publications
  - (9) Motor Truck Regulations

American Concrete Institute
- S–6, on Concrete Roads and Pavements
- E–3, on Research
- E–5, on Aggregates
- S–2, on Reinforced Concrete Highway Bridges and Culverts
- C–5, on Central Mixing and Proportioning Plants
- C–6, on Field Methods

American Road Builders Association

American Society of Civil Engineers
- Committee on Soils, with reference to their bearing value
- Committee on Highway Engineering
- Committee on Contract Standard Clauses
- Committee on Research

American Society for Municipal Improvements
- (1) Street Paving and Street Design, Street Maintenance, and Street Railway Construction
- (2) Traffic and Transportation
- Eleven Committees on Specifications

American Society for Testing Materials
- Committee D–4, on Road and Paving Materials
  - Non-bituminous materials
  - Bituminous materials
- Committee C–1, on Cement
- Committee C–3, on Brick
- Committee C–6, on Drain Tile
- Committee C–9, on Aggregates

Bureau of Public Roads, U. S. Department of Agriculture

Bureau of Standards, U. S. Department of Commerce

Eno Foundation

National Automobile Chamber of Commerce

National Highway Traffic Association
- Committees on
  1. Uniform Highway Signs
  2. Traffic Capacity and Widths of Highways Outside of Communities
  3. Status of Construction of Highway Curves and Recommended Practice to Increase the Safety of Traffic
  4. Regulations Covering Speeds, Weights, and Dimensions of Motor Trucks and Trailers
  5. Highway Transport Franchises
  6. Highway Transport Clearing Houses

National Safety Council
- Public Safety Division

Society of Automotive Engineers
- Highways Committee
- Research Committee

The functions of Committees may be conveniently divided into the following:

I. To furnish information from statistical or experimental data, e.g., Committees of Advisory Board on Highway Research; Committee on Track, American Society of Civil Engineers and the American Railway Engineering Association.

II. To formulate standards, e.g., Committees of American Society for Testing Materials, and of American Association of State Highway Officials.

III. To recommend methods and practice, e.g., Joint Committee on Concrete and Reinforced Concrete, Committee of American Railway Engineers Association.

IV. To consider and recommend policies, e.g., Committee of Chamber of Commerce of the United States.

V. To deal with education, and Highway Education Board.

VI. Coordinating, e.g., Advisory Board on Highway Research.

An analytical study of committees and function of the organizations now working in the field of highway research and highway practice would no doubt be helpful.

PART VII

INDEX: ORGANIZATIONS ALPHABETICALLY BY STATES*

*Numbers refer to pages.

Bulletin of the National Research Council

Volume 1

Number 1. The national importance of scientific and industrial research. By George Ellery Hale and others. October, 1919. Pages 43. Price $0.50.

Number 2. Research laboratories in industrial establishments of the United States of America. Compiled by Alfred D. Flinn. March, 1920. Pages 85. Price $1.00. [Out of print. See Number 16.]

Number 3. Periodical bibliographies and abstracts for the scientific and technological journals of the world. Compiled by Ruth Cobb. June, 1920. Pages 24. Price $0.40.

Number 4. North American forest research. Compiled by the Committee on American Forest Research, Society of American Foresters. August, 1920. Pages 146. Price $2.00.

Number 5. The quantum theory. By Edwin Plimpton Adams. October, 1920. Pages 81. Price $1.00. [Out of print.]

Number 6. Data relating to X-ray spectra. By William Duane. November, 1920. Pages 26. Price $0.50.

Number 7. Intensity of emission of X-rays and their reflection from crystals. By Bergen Davis. Problems of X-ray emission. By David L. Webster. December, 1920. Pages 47. Price $0.60.

Number 8. Intellectual and educational status of the medical profession as represented in the United States Army. By Margaret V. Cobb and Robert M. Yerkes. February, 1921. Pages 76. Price $1.00.

Volume 2

Number 9. Funds available in 1920 in the United States of America for the encouragement of scientific research. Compiled by Callie Hull. March, 1921. Pages 81. Price $1.00.

Number 10. Report on photo-electricity including ionizing and radiating potentials and related effects. By Arthur Llewelyn Hughes. April, 1921. Pages 87. Price $1.00.

Number 11. The scale of the universe. Part I by Harlow Shapley. Part II by Heber D. Curtis. May, 1921. Pages 47. Price $0.60.

Number 12. Coöperative experiments upon the protein requirements for the growth of cattle. First report of the Subcommittee on Protein Metabolism in Animal Feeding. By Henry Prentiss Armsby, Chairman. June, 1921. Pages 70. Price $1.00.

Number 13. The research activities of departments of the State government of California in relation to the movement for reorganization. By James R. Douglas. June, 1921. Pages 46. Price $0.60.

Number 14. A general survey of the present status of the atomic structure problem. Report of the Committee on Atomic Structure of the National Research Council. By David L. Webster and Leigh Page. July, 1921. Pages 61. Price $0.75.

Number 15. A list of seismologic stations of the world. Compiled by Harry O. Wood. July, 1921. Pages 142. Price $2.00.

Vol. 4, Part 4 NOVEMBER, 1922 Number 22

BULLETIN

OF THE

NATIONAL RESEARCH COUNCIL

Mechanical Aids for the Classification of American Investigators, with Illustrations in the Field of Psychology

BY

HAROLD C. BINGHAM

ASSISTANT DIRECTOR

Research Information Service, National Research Council

PUBLISHED BY THE NATIONAL RESEARCH COUNCIL

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THE NATIONAL ACADEMY OF SCIENCES

WASHINGTON, D. C.

1922

Announcement Concerning Publications
of the
National Research Council

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To Committee on Publications and Publicity,
National Research Council.

As Chairman of the Research Information Service I desire to recommend to the Committee on Publications and Publicity the issuance as Bulletin of the National Research Council of a report entitled "Mechanical aids for the classification of American investigators with illustrations in the field of psychology," by Harold C. Bingham.

This report describes with minute care the personnel system of the Research Information Service and the Findex classification which has been developed for convenient use of the system. It illustrates various uses of the personnel file by presenting results pertaining to the characteristics, interests, and research activities of American psychologists. It is believed that the report will prove especially valuable to persons who are in need of mechanical sorting devices and that the illustrative materials will acquaint American psychologists with many facts not previously available which are of obvious importance for the further development of science.

It is earnestly hoped that readers of Mr. Bingham's report will avail themselves as they have need, either through correspondence or by visit, of the invaluable information concerning American investigators and their work which is immediately accessible in the personnel file of the Research Information Service.

Respectfully submitted,

ROBERT M. YERKES, *Chairman,*
Research Information Service.

June 21, 1922.

Vol. 4, Part 4 November, 1922 Number 22

BULLETIN

OF THE

NATIONAL RESEARCH COUNCIL

MECHANICAL AIDS FOR THE CLASSIFICATION OF AMERICAN INVESTIGATORS, WITH ILLUSTRATIONS IN THE FIELD OF PSYCHOLOGY

BY HAROLD C. BINGHAM

Assistant Director, Research Information Service,
National Research Council

CONTENTS

PREFACE

Since its organization the Research Information Service of the National Research Council has sought and developed fundamental apparatus for the benefit of science and technology. Economical methods in the location of available knowledge and in the distribution of appropriate information have received special consideration. Within these activities the guiding policy has been to build clearing-house machinery without establishing an unwieldy store house. To that end numerous reference lists, catalogues, and compilations of sources have been prepared so that today the Service is equipped with certain mechanisms in working operation. This machinery exists for the benefit of investigators who wish to avoid duplication of effort and waste of energies.

The present report is based on a successful experiment in this informational service. By means of a punch-card system, personnel records, as reported by a homogeneous group of American scientists, have been handled as keys to sources of knowledge. On the strength of this demonstration of practicability and serviceability, steps have been taken to apply the same procedure to the personnel records of all American investigators, as well as to those who are qualified for research, in the natural sciences and corresponding technologies. It is contemplated that this shall be the first of a series of reports on the research personnel of various fields of science and technology.

For research workers in fields other than psychology, subsequent reports of results will be undertaken similar to those presented under section 9, page 21, on the activities of American psychologists. On this account the present paper places special emphasis upon the method and technique underlying the compilation of such results. Although this introductory report will have particular significance for psychologists whose records were used in the experiment, it is believed that the general treatment of mechanical aids in handling such records will be of interest to a wide circle of readers.

In certain respects the Research Information Service is a novel organization. It is favorably located within an institution that democratically represents the scientific and technological people of the country. The organizers of the National Research Council conceived the Service as a clearing-house of scientific information for these same people. It is hoped that a description of a single cogwheel in this clearing-house machinery will help to bring about, among scientific men, a clearer understanding of the activities and an appreciation of the resources which they are invited to use.

Washington, D. C.,
October 5, 1922.

HAROLD C. BINGHAM.

Part I

MECHANICAL SYSTEMS

1. The personnel mechanism.

In an informational clearing-house for science and technology, a file of research scientists should provide immediate reference to sources. Theoretically, every request for information can be hopefully referred for answer to a well developed personnel file because *people* have command of the known facts. Invariably, it is a *person* who publishes on a given subject, who reports a discovery, who invents an instrument, or who develops a method. It is *someone* working in a given field, who is prepared to give information about a particular project, or who is qualified to advise about a specific question. The research of a laboratory, the development of its facilities, the choice of problems for investigation—in short, the policy and trend of research in any institution is reflected in the interests and activities of its individual investigators. Personnel records, if complete, contain the answers to inquiries about current research. They also provide a ready source of information about publications and thus supplement bibliographic compilations.

To fulfil its maximum service, however, the personnel catalogue must be so organized that records of the activities of the proper people are immediately available when needed. With a classified mechanical file carrying information furnished by original personnel records, one can on a moment's notice command pertinent information about any question falling within the scope of the activities recorded. Moreover, with the original records alphabetically filed, information about any individual can be promptly produced. Thus, the records of the mechanical file and the alphabetical file supplement one another; and jointly they cover the whole range of probable requests for information.

2. Selection of a mechanical aid for handling personnel records.

The need for a mechanical file having been recognized, active work upon its development was started in 1921. Previously, in 1920, investigation of available systems had been started by the Chairman of the Research Information Service, when some 9,500 biographical sketches were obtained in co-operation with the editors of the third edition of American Men of Science. In 1921 the total number of personnel records was independently increased by the Research Information Service to approximately 11,000; and in 1922 the total number reached nearly 14,000.

Special consideration was given to the selection of a suitable mechanical system for handling these personnel records. Careful attention was given to various mechanical devices, selling arguments and advertising features were studied, and advice from experienced users was obtained.[1] Different visible indexes, card-tab systems, and punch-card systems were considered with a view to their limitations and flexibility. The problem was also considered from the standpoint of both initial and subsequent expenditure without overlooking the prospective needs of the various mechanisms of the Research Information Service. Moreover, a mechanical system was sought by means of which selected groups of investigators or of subjects could be obtained for preliminary use in the collection of information on particular topics.[2]

For safe filing of original personnel records, a straight alphabetical arrangement seemed essential. It was desirable, therefore, that the mechanical system selected should provide for all phases of personnel classification including both systematic arrangement of duplicate records and selections in accordance with special informational demands. It should not only furnish ready reference to an individual or to a group of investigators, and thus to special fields of inquiry, but it should also eliminate laborious inspection and hand manipulation of records in locating various kinds of information.

Two files were thus contemplated and are now in process of development. The one is consulted for information about an individual investigator; the other is used to obtain lists of investigators having prescribed qualifications. As a basic skeleton, fifteen major groups

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[1]Professor Edwin G. Boring of Harvard University, who planned and supervised the analysis of nearly 1,750,000 psychological records in the U. S. Army and Professor Donald G. Paterson of the University of Minnesota, who was formerly employed by the Scott Company in analyzing personnel records for occupational groups, generously advised about the relative merits of different punch-card methods. Professor Boring's statistical work was based on Hollerith analyses; Professor Paterson had used the Powers system. Mr. Harrison E. Howe, Editor of the Journal of Industrial and Engineering Chemistry, also made valuable suggestions on the basis of former experience in punch-card analyses.

[2]Mr. J. David Thompson, Scientific Associate, Research Information Service, has recently suggested the use of a mechanical system for locating special materials required in industry or science. Materials with any desired properties might thus be readily found which would satisfy actual conditions in manufacturing processes, such as resistance to certain kinds of wear and tear or reduction in cost of production. This is only another of the various mechanisms which have been suggested, contemplated, or initiated for use in the Service.

of science and technology have been tentatively adopted according to which the records of the mechanical file are or will be arranged.[1] Subdivisions of any group, additions of more, or consolidation of groups will be made as experience indicates necessity.

The personnel record for each investigator, without regard to the group in which he is classified, is entered on a form like that in Figure 1. Because investigators frequently are active in more than one field of research, the necessity of filing these original records in a single alphabetical file is further apparent. The classified mechanical file is thus required to furnish all cross-references. Any mechanical selection of names may be checked in final revision against the file of original records.

It was estimated at the outset of this undertaking that the total number of investigators belonging to any single group would not exceed four thousand, and in certain groups the number would not exceed five hundred. Later subdivision of the largest groups into smaller classes—*e. g.*, engineering into civil, chemical, mechanical, and so forth—greatly reduced the maximum number of records appearing in a single group. The total number of research people in the personnel file at that time was approximately 11,000 and it was estimated that for several years to come, this total would not exceed 20,000. Always, then, the selection of a list of investigators meeting certain specifications would be made from a single group, sometimes repeated with one or more groups, containing but a few hundred or at most several hundred records. Because the possibility of subdividing a group always remains, it was decided that the adoption of a mechanical aid should be based on sorting requirements for less than one thousand and averaging perhaps six or seven hundred records.

3. Visible indexes and card tabs.

The essential requirement for the personnel project, then, was a means of producing, mechanically—that is, without hand manipulation or personal inspection—the records of groups of people having certain definite and prescribed qualifications. Preliminary study indicated that visible indexes alone would provide little, if any, advantage over the file of original records in making this personnel

[1]These groups comprise the following:

1 Agriculture	6 Economics and Statistics	11 Mathematics
2 Animal Biology	7 Education	12 Medicine
3 Anthropology	8 Engineering	13 Plant Biology
4 Astronomy	9 Geography	14 Physics
5 Chemistry	10 Geology	15 Psychology

information available. Visible indexes for the most part provide merely for economical reference to cards already in file. However convenient such devices may be for making information visible and

Name Surname Christian name

Address:

Position:

Inst. or firm:

Date of appointment:

Per cent time for research:

Previous positions: Date
Title Inst. or firm From To

Nationality Place of birth Date

Married Name of wife or husband Date

Degrees, with insts. and dates:

If Ph. D. state
Major subject.

Minor subjects:

Thesis topic:

Place and date of publ. of thesis:

Leading science teachers:

Membership in societies:

Lectureships:

Other distinctions or honors:

General field of research:

Special research interests:

Investigations in progress:

Principal publications (arranged chronologically, with title, place and date):

Statement of special qualifications, achievements, plans or preferences:

NAME

DATE

SERIAL NO.

FIGURE 1. Blank form used in collecting and filing information about American investigators. Reduced from the original which is 8½ inches by 11 inches.

records accessible, they do not always function mechanically in producing records with multiple specifications. They do make visible a limited number of specifications, such as name, address, occupation, age, and so forth; but the usableness of such methods in providing combinations of specifications depends not only upon the classified arrangement of the records but also upon the accuracy of the person operating the file. This method, too, would often require duplication of original personnel records because the frequent tendency of investigators to distribute their research activities among two or more fields, makes cross-referencing necessary. Thus the visible index method was rejected as unsuited to the particular project.

Possibilities in flagging devices also received consideration. Appropriate tabbing of the original records would, of course, make possible the preservation of the desired alphabetical arrangement. Obviously, however, the card tabs would provide for but few specifications beyond those of the fifteen major groups listed on page 7. Competent advice, too, indicated that not more than twenty classifications by means of card tabs is practicable under any circumstances. Arrangement of duplicate personnel records into major groups within which card tabs might indicate subdivisions was also considered; but even with this arrangement the number of recorded specifications would be too limited for the personnel project. Moreover, any conceivable adaptation of visible indexes and card tabs would, in order to reveal more than a limited combination of specifications, require both hand manipulation and conscious selection.

Neither of these methods, therefore, would furnish a means of finding, without hand manipulation or personal inspection, the records of individuals having definite, prescribed specialties. The selection of a mechanical aid seemed clearly to point to the choice of some suitable punch-card system.

4. Hollerith and Powers systems.

Three punch-card systems were carefully considered—Hollerith, Powers, and Findex. It was necessary merely to choose the one best adapted to the particular needs of the personnel project. It was deemed unnecessary to establish the relative merits of the three systems, for it was recognized that each system has certain distinctive features which may be highly useful in one project, but relatively useless in another. Therefore, the choice eventually made cannot possibly be construed as a reflection upon either of the systems rejected.

So far as the problems of the personnel project are concerned, it was seen that the Hollerith and Powers systems might be treated as a single system. The particular features distinguishing them, such as, for example, the automatic counting device, though valuable for some purposes, were of slight significance for this project. The requests which are received in the Research Information Service most frequently contain the question "who"; very rarely, "how many." For such quantitative demands as do arise, the numbers are regularly small and may be readily counted. But in such essential features as form of card, method of recording information, and method of sorting cards, the Hollerith and Powers systems commanded serious consideration.

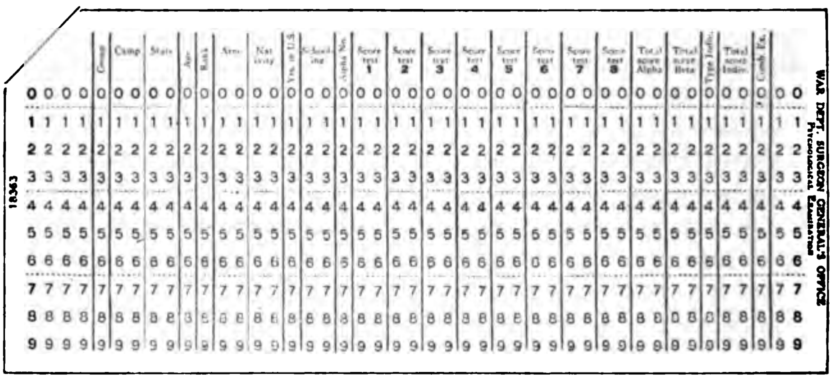

FIGURE 2. Record card used in mechanical sorting by Hollerith system. All information is coded in numerical terms and holes are punched in the corresponding numbers on the card. A two-digit number is represented by two holes in adjacent columns. The figure is reduced from 7⅜ inches long. Reprinted from Psychological Examining in the United States Army, *National Academy of Sciences, Memoirs 15*, 1921, page 568.

An unused Hollerith or Powers card, unpunched, appears as in Figure 2. The face of the standardized card contains ten rows of numbers which are divided vertically into columns. To each number there is assigned a specific definition in a pre-established code. The procedure in transferring facts from an original record to one of these punch cards consists in making a hole, by means of an accurate punching mechanism, through the proper number in each column. An item coded by a two-digit number is recorded on the mechanical card by punching in the first column the number representing the tens digit; in the second column, the number representing the units digit. Items for which three or more digits are required may be similarly recorded. In this manner a card is prepared for each individual record.

After a punch card has been prepared for each original record, mechanical selection may be accomplished by means of an electrically driven tabulating machine. A sorting device is set to select all cards on which a given code number has been punched. The cards thus selected are mechanically carried to a receptacle while the rejected cards (pertinent item not punched) are deposited in another receptacle. Cards for individuals meeting a given specification are thus brought together in one group. If multiple specifications are to be represented by each individual, it is necessary to make successive sorts of the cards until each specified item has entered into the final selection.

Each selection by means of the Hollerith or Powers systems, it is clear, will disarrange any previous classification which may have been adopted for the cards. To return the sorted cards to the prearranged classification, provision must be made in the code in order that the mechanical sorting process may be used to rearrange the cards in their proper filing order. This requirement was obviously impracticable where the desired file arrangement was that of systematic groups of records, each alphabetically arranged.

Another reason for rejecting the Hollerith and Powers systems for this project lay in the continuous overhead expense that would be assumed by the rental of a sorting machine. Tabulating machines for the Hollerith and Powers cards are not offered for sale. On the probability that, for a long time to come, the machine would be idle much of the time, but urgently and promptly needed for short intervals, it was deemed inadvisable to incur the rental expense or to rely upon sorting facilities outside of the Research Information Service.

Furthermore, advice of experts indicated that it would be hazardous to dispense entirely with the tabulating machine and to rely upon hand-needling of cards. Various estimates by these advisors placed the maximum number of cards that can successfully be hand-needled well under the smallest number of personnel records that must be handled in the Research Information Service. Under conditions of work where efficiency has been obtained through continual practice, it was reported that a tabulating machine is essential for more than 10,000 cards. On the other hand, when less than 500 cards are intermittently classified, it was reported that the usefulness of the machine is questionable. On the basis of such facts, it seemed undesirable to install Hollerith or Powers systems without adequate provision for mechanical sorting.

Storage requirements furnished an additional reason for rejecting the Hollerith or Powers punch cards. To prevent curling, these cards must be stored compactly when not in use. This precaution has been found quite necessary to eliminate the annoyance of "jamming" in the tabulating machine, caused by cards that do not lie quite smooth and flat. This storage requirement could not be harmonized with the plan of using the mechanical file for subject classification of records to which one might turn for names and addresses as well as to the alphabetical file of original records.

5. The Findex system.

An unused Findex card differs in several respects from Hollerith or Powers cards. Each position on the Findex card is marked not only by a number but also by a round hole. The number of these holes varies with the size of the standard card which is adopted. As few as 40 and as many as 180 positions on cards of appropriate size have been adopted by different users of this system. The actual range of possible positions is from 1 to 186. The illustration presented in Figure 3 is reduced from a Findex card 8 inches by 6 inches, having 130 positions.

As in the systems previously described, each numbered position is defined in a pre-established code. Figure 3 shows a Findex card in which certain numbers representing one person's qualifications have been stroked. The facts about each individual are finally recorded on one of these cards by means of a vertical slot mechanically punched through the stroked number (cf. Figures 3 and 4). Each coded item is thus represented by a single slot-punch. A description of the details in this process appears in section 6 where a Findex experiment with personnel records is described.

Briefly, the Findex procedure consists in filing these slotted cards loosely in drawer-like containers where they are held in position by means of removable rods passing through the six marginal slots, exclusive of the short slot at the middle of the top margin, shown in Figure 3. The guide rods terminate in the ends of the container. See Figure 5, facing page 16. When it is necessary to select a special group of cards a rod is inserted for each specification. Then, by inverting the container the cards slotted for all specifications that have been rodded, drop down one-half inch, revealing the names of the persons who meet the requirements. The names may then be listed or the cards may be counted in accordance with the nature of the demand. If individual reference to the cards is desirable, these guide rods may be withdrawn from the marginal slots leaving the cards removable from the container.

For the requirements of the personnel project in the Research Information Service, it seemed probable that the Findex system might be particularly useful. Accordingly it was decided to give it experimental trial with a single group of scientists, and with a group small enough to make the experiment controllable, but large enough to give the system a fair trial. Not only because the number of American psychologists (approximately 500) was satisfactory, but also because of the writer's professional acquaintance in that field, this group was

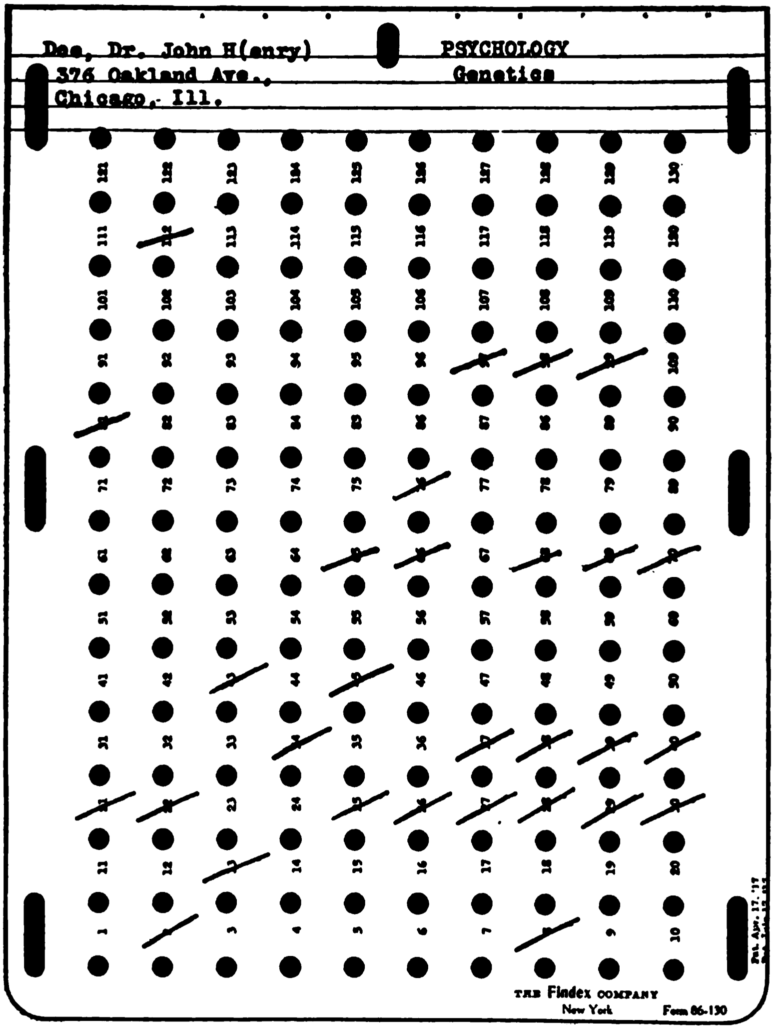

FIGURE 3. Stroked Findex card for an American investigator. At the top of the card the name, address, and other variable information are regularly typed. See Figure 4, page 16, for a further stage in the preparation of this record. Both figures are reduced from cards 8 inches long.

selected for the experiment. Further explanation of the Findex method will be found in connection with the discussion of the actual experiment which follows.

Part II.

APPLICATION OF FINDEX SYSTEM IN THE FIELD OF PSYCHOLOGY

6. An experiment in findexing American psychologists.

Throughout the experiment with American psychologists, Findex cards containing provision for 120 classifications were used. At the conclusion of the experiment, the number of positions on the Findex cards was increased to provide for 130 classifications.[1] It was decided to make the first 30 classifications uniform for all scientific groups. Thus for any science, the stroking of number 13 indicates birth during the decade from 1860 to 1870. But each number above 30 represents a different specialty in each personnel group for which a separate code exists. For example, number 65, stroked in accordance with the code for psychology (page 18), means that the individual has reported special work in abnormal psychology. In chemistry, the same number represents specialization on chlorine; in plant biology, the specialty represented is mutations.

Figure 4 shows the slot-punched Findex record of the same person whose record is stroked in Figure 3. Aside from a negligible number of fictitious items, the illustration is a reproduction of the record of an American psychologist who has a subordinate interest in genetics. When a code for genetics has been prepared, a blue Findex card will appear in the two groups of psychologists and geneticists, and each will be slot-punched in accordance with the appropriate code. The blue card indicates that the individual is interested in more than one of the main subdivisions by which the cards are filed and must, therefore, have two cards.

The scientific or technological classifications represented on the slot-punched card refer both to past and current activities. In the preparation of the cards, publications, investigations in progress, research interests, former and present positions are considered in order that every possible source of personal information may come out when a mechanical sort is made.

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[1]This increase in positions was adopted because it provided for ten additional classifications without increasing the size of the card. It utilized to excellent advantage space on the card that had been largely wasted during the experiment.

The slotted numbers shown in Figure 4 refer to the following items:

2	college	39	2 years (experience)
8	institution	40	promotion
13	1870 (birth)	43+45	Maine (birth)
21	administrator	65	abnormal
22	teacher	66	applied
25	consultant	68	experimental
26	married	69	genetic
27	doctor's degree	70	group
28	doctor of medicine	76	biology
29	recognition	81	hygiene
30	publications	97	intelligence levels
34	1918 (appointment)	98	inheritance
37	10 years (experience)	99	modifiability
38	5 years (experience)	112	pathology

For definition of the items listed, see the psychology code as presented on pages 18-21.

Figure 5 shows a container full of cards which have been stroked and slotted. Six rods have been inserted for a selection of individuals who qualify with respect to a combination of six specifications.[1] By inverting the container, only those cards which have been slotted for each of these six specifications can fall. When the container is returned to its upright position the locking rod, inserted as indicated at the top of the box, will hold this group of cards in place. In Figure 5 the selection has been completed and the desired cards stand out for convenient inspection. As shown in the figure, the names on the cards of this selected group are visible, whereas the names on the other cards remain relatively invisible.

Figure 6 shows the container inverted allowing the cards, which have been slotted in accordance with the rodded specifications, to fall down. After all of the cards bearing these six specifications have dropped down, the locking rod is inserted and the container is returned to its upright position, as shown in Figure 5, for inspection of the names on the different cards. This is the positive Findex method. It can also be used negatively by first inverting the container and then inserting the sorting rods for the specifications which are not wanted. When the rodded container is returned to its upright position, the cards with the negative specifications fall down (out of

[1]Reference is not made to the psychology code in explanation of this figure because the combination of specialties marked by the sorting rods, shown in position, represents a group of classifications which does not belong to the field of psychology.

sight) and the cards which do not possess the negative specifications stand up for inspection. The working of the system itself is extremely simple.

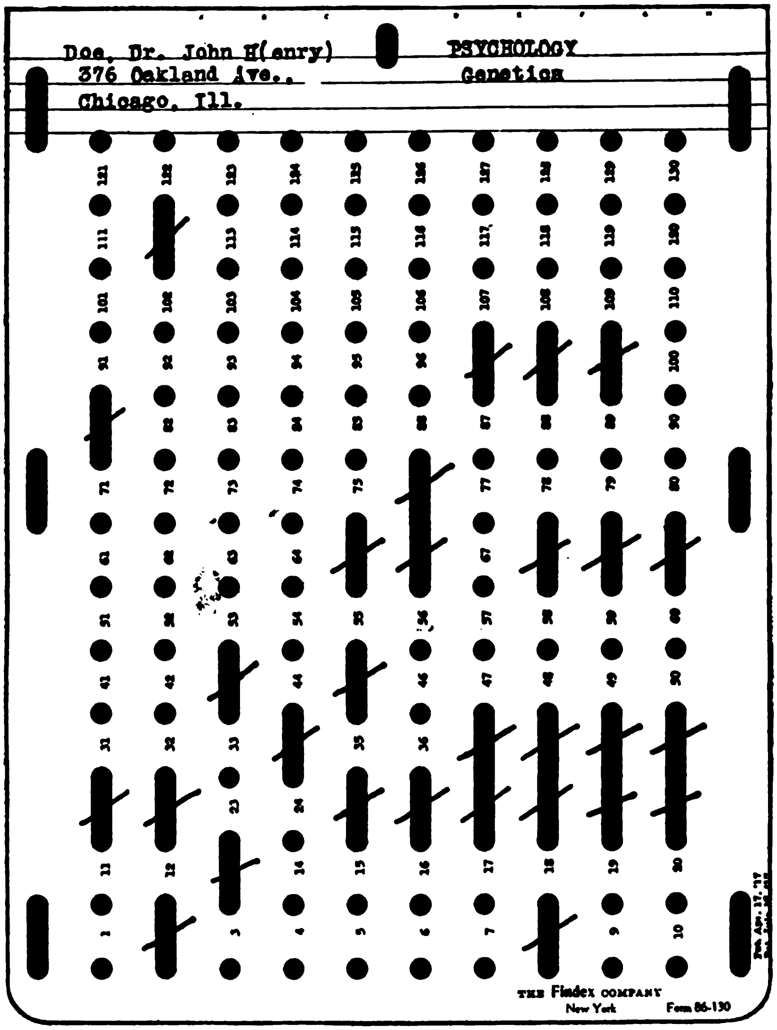

FIGURE 4. Slot-punched Findex card. This same card, stroked, but not slotted, is shown in Figure 3. If a small rod is inserted in one of the round holes, the card will be held fast when the container is turned upside down; but if the rod passes through one of the half-inch slots, the card will drop down one-half inch when the container is inverted.

The real task in findexing American psychologists arose in connection with the collection of complete original records and in the adop-

Figure 5. Findex container showing guide rods and sorting rods. The containers used in the Research Information Service are slightly less than 12 inches long and a little more than 6 inches wide, inside measurements. They hold approximately 600 cards. Brass plates about 1/16 inch thick containing holes for proper alignment of the guide and sorting rods are securely fastened by screws to each side of both the face and the partition. A similar plate is attached to the inside of the rear wall of the container.

FIGURE 6. Findex container in sorting position. This stage in the sorting process actually precedes the one illustrated in Figure 5. After the cards having the desired specifications have dropped down, a locking rod is inserted through the slot at the top of the inverted container. The locking rod passes through the short slot of the cards as shown at the top of Figures 3 and 4. In this illustration it has not yet been inserted. In Figure 5 the locking rod is shown in position.

tion of a satisfactory classification for use in the code. It was necessary to prepare a classification not too detailed and yet capable of furnishing various kinds of specific information. In an effort to obtain a terminology that would not only provide for various terms used by different psychologists but would also cover the probable demands of querists in need of specific information, three sources were chiefly used: (1) the subjects of research reported in the yearbook (1920) of the American Psychological Association; (2) the subjects of research reported on the original personnel records; and (3) the specifications enumerated in a collection of actual requests from people desiring to fill psychological positions.

7. Making the Findex code for American psychologists.

On the basis of these criteria the following code for use in classifying American psychologists was finally adopted. In the course of working out this trial code numerous imperfections appeared; but it was not practicable to eliminate them during the course of the experiment. Subsequent efforts have been directed toward the elimination of similar shortcomings in codes for other personnel groups.[1] As the adoption of a method for handling all records of American scientists and research technologists was awaiting the outcome of this practical trial, no further energy has yet been expended on the perfection of the psychology code. To develop an ideal experimental technique before launching the psychological experiment was neither expedient nor possible. On the contrary, it was necessary to treat this experiment as a preliminary trial, the results of which would determine its general applicability for the other sciences and technologies.

The psychology code represents a continual development throughout the period of the experiment. Definitions of terms frequently had to be enlarged to take care of new items for which provision had not originally been made. Occasionally, new topics had to be added

[1]Codes for handling various personnel groups have been prepared or are in preparation (1) for agriculture, animal biology, and plant biology by Miss Ruth Cobb, formerly Scientific Associate, Research Information Service; (2) for chemistry by Doctor C. J. West, Secretary, Division of Chemistry and Chemical Technology, and formerly Scientific Associate, Research Information Service; (3) for engineering by Mr. H. F. Whittaker, Scientific Associate, Research Information Service, with preliminary assistance from his predecessor, Captain Walter Graham; (4) for geography by Doctor N. M. Fenneman, Chairman, Division of Geology and Geography; (5) for geology by Doctor H. P. Little, formerly Executive Secretary, Division of Geology and Geography, and Scientific Associate, Research Information Service; (6) for physics by Doctor W. E. Tisdale, Executive Secretary, Division of Physical Sciences, and Scientific Associate, Research Information Service. Advice and assistance has also been received from many other individuals and organizations.

when it was not practicable to provide for them in revised definitions of other topics. Moreover, such changes had to be made without disturbing the skeleton which had been adopted at the outset. As a result, the code is not so logically arranged as it will be when revised It is not presented here in order to recommend its perfection of form but merely as a record by means of which the results to be outlined in the rest of the report may be interpreted. Whenever a change was made in the code due caution had to be exercised to revise the previously findexed records. Changes made on original records too had to be systematically made on the corresponding Findex cards.

8. The psychology code.

NAME, ADDRESS, AND FIELD OF RESEARCH

Type at top of Findex card. Enter surname first, insert comma, and give *full* Christian name. Enclose within parentheses portions of Christian names ordinarily omitted in correspondence.

PROFESSIONAL EXPERIENCE

Make appropriate Findex record for any institution with which past or present professional experience is reported. Experience following arrival at professional level only will be recorded.

1 University—Include instructors and others having equal or higher rank.
2 College—Include instructors and others having equal or higher rank.
3 Preparatory school—Include normal schools, if not of collegiate standing, and special schools approximating Junior College.
4 Grade school—Include private and special schools below high school in which work of a psychological nature has been done.
5 Commercial concern—e. g., advertising and selling agencies.
6 Industrial concern—e. g., production and manufacturing establishments.
7 Federal, state, or city affairs—Include professional psychologists in army.
8 Institution—Other than those enumerated above, such as museums, hospitals, churches, reformatories, criminal courts, etc.
9 Private—Independent work in connection with which remuneration is not received as salary from an institution.
10 Woman—Negative sort will produce male psychologists.

BIRTH

Year in or before which born; e. g., 15 means born during the period 1876-1880, inclusive.

11 ——
12 1860.
13 1870.
14 1875.
15 1880.
16 1885.
17 1890.
18 1895.
19 1900.
20 ——

NOTE: In subsequent classifications, birth periods have been coded to make decades begin with years ending in zero, and half decades with years ending in zero or five. Positions 11 and 20 may be used as necessary. For example, 20 may be used for a birth period represented by 1905; 11, by 1910. When 1915 must be added, it is assumed that 1860 will no longer be required. Provision is thus made for a continuous cycle.

PROFESSIONAL STATUS

Make appropriate Findex records for any item which is or has been applicable throughout the period of professional activity.

21 Administrator—executive, director, etc.
22 Teacher.
23 ——
24 ——
25 Consultant—e. g., in therapeutic, educational, or industrial problems.
26 Married—Include widow or widower.
27 Doctor's degree—Ph. D., Sc. D., etc.; exclude M. D.
28 Doctor of Medicine—M. D.
29 Recognition—Membership in clubs and societies; include only those which represent professional distinction.
30 Publications—Make appropriate Findex record for one contribution or more to an important scientific periodical.

DATE OF PRESENT APPOINTMENT

Year in or before which appointed.

31 1900.
32 1910.
33 1915.
34 1918.
35 1920.
36 1922.

NOTE: In subsequent codes these items have been omitted.

TOTAL EXPERIENCE

37 10 years or more—Include also 38 and 39.
38 5 years or more—Include also 39.
39 2 years or more.
40 Promotion—Include only when promotion has followed arrival at professional level.

NATIVITY

a. United States

41-42	Alabama	43-45	Maine	46-48	Oregon
41-43	Alaska	43-46	Maryland	46-49	Pennsylvania
41-44	Arizona	43-47	Massachusetts	46-50	Rhode Island
41-45	Arkansas	43-48	Michigan	47-48	South Carolina
41-46	California	43-49	Minnesota	47-49	South Dakota
41-47	Colorado	43-50	Mississippi	47-50	Tennessee
41-48	Connecticut	44-45	Missouri	48-49	Texas
41-49	Delaware	44-46	Montana	48-50	Utah
41-50	Dist. of Col.	44-47	Nebraska	49-50	Vermont
42-43	Florida	44-48	Nevada	51-52	Virginia
42-44	Georgia	44-49	New Hampshire	51-53	Washington
42-45	Idaho	44-50	New Jersey	51-54	West Virginia
42-46	Illinois	45-46	New Mexico	51-55	Wisconsin
42-47	Indiana	45-47	New York	51-56	Wyoming
42-48	Iowa	45-48	North Carolina	51-57	Panama
42-49	Kansas	45-49	North Dakota	51-58	Porto Rico
42-50	Kentucky	45-50	Ohio	51-59	Philippines
43-44	Louisiana	46-47	Oklahoma	51-60	Hawaii.

b. Foreign

52-53	Armenia	52-57	Bulgaria	53-54	Cuba
52-54	Austria	52-58	Canada	53-55	Denmark
52-55	Belgium	52-59	Central Amer.	53-56	Egypt
52-56	Bohemia	52-60	China	53-57	England

b. Foreign (Continued)

53-58 Finland
53-59 France
53-60 Germany
54-55 Greece
54-56 Holland
54-57 Hungary
54-58 India
54-59 Ireland
54-60 Italy
55-56 Japan
55-57 Mexico
55-58 Norway
55-59 Poland
55-60 Roumania
56-57 Russia
56-58 Scotland
56-59 Serbia
56-60 South Africa
57-58 South America
57-59 Spain
57-60 Sweden
58-59 Switzerland
58-60 Turkey
59-60 Other Countries.

Geographical Interests

Not recorded in Findex file for American psychologists. For certain scientific and technological groups, such records are important.

Distinctions or Honors

61 As student—Important fellowships, scholarships, etc.
62 In profession—Prizes; listing in Who's Who in America; important institutional connections such as academic or federal responsibilities; director of important educational surveys, etc.
63 "Starred" in American Men of Science. Member of American Academy of Arts and Sciences, American Philosophical Society, or National Academy of Sciences.

Special Activities

The following items will be included in the Findex record when reported as a present or a past activity and when publications or professional reputation warrant. Give consideration to such items as academic titles, professonal positions, doctorate majors and minors, honors and recognition, and reports of research activities.

64 Educational—Educational psychology in distinction from education (78) which appears under related fields.
65 Abnormal—Include all activities concerning atypical behavior, also such other topics as dreams and complexes; exclude activities concerning sub- and super-normality.
66 Applied—As reported in any field; e. g., law, industry, education, etc.
67 Differential—Individual characteristics.
68 Experimental—Laboratory and field technique and investigations; include development and standardization of tests; exclude mere users of tests.
69 Genetic—Development in the individual or race.
70 Group—Social and racial characteristics.
71 General—Psychology of the normal, adult, human individual; unqualified reports of research activities.
72 History—Of the science or phases thereof.
73 Infra-human—Animals and plants; include what is frequently termed comparative psychology.
74 Mental tests—Include both achievement and capacity tests.

Related Fields

To be recorded on Findex card as reported or as indicated by publications and experience; include majors and minors for doctorate.

75 Anthropology
76 Biology
77 Business
78 Education
79 Esthetics
80 Ethics
81 Hygiene
82 Industry

83 Law
84 Logic
85 Medicine
86 Philosophy
87 Physiology
88 Religion
89 Sociology
90 Statistics
91 ——
92 ——
93 ——
94 ——

Other Interests and Specialties

This group largely includes topics for which special need has developed since the Findex code was originally established. Unused numbers may be assigned as need for additional classifications arises.

95 Clinical.
96 ——
97 Intelligence levels—Feeblemindedness, mental deficiency, superiority, etc.
98 Inheritance—Racial and individual.
99 Modifiability—Learning.
100 Objective—Quantitative methods.
101 Psychics—Psychical research.
102 Psycho-analysis.
103 Psycho-biology.
104 Psycho-physics.
105 School subjects.
106 Subjective methods.
107 Systematic—Descriptive treatises; text books, etc.
108 Temperament.
109 Theoretical.
110 ——
111 Occupational fitness—Includes vocational and trade studies.
112 Pathology.
113 Sex.
114 ——
115 ——
116 ——
117 ——
118 ——
119 ——
120 ——

9. Activities of American psychologists.

In the past a few attempts have been made to record the status, or to review the history, of psychology. These efforts have usually taken the form of reports dealing with the degree of specialization and activity by means of periodic analyses and classifications of the literature, or by means of surveys of departmental and laboratory organizations in academic institutions. The Findex method of recording trends of psychology contrasts notably with these former methods and on such grounds as economy, thoroughness, and adaptability it is more promising than the earlier procedure. As in some of the physical sciences, so in psychology, there may be tendencies not adequately revealed by the usual academic standards of publication, association membership, laboratory development, and so forth. A survey of personal activities offers one of the most reliable methods for outlining the course or describing the status of any movement.

In addition to its usefulness in furnishing information about various subjects of psychological inquiry, the Findex punch-card system has proved valuable in reviewing different trends of professional activity. The data which are presented in this section concerning the activities of psychologists illustrate some of the varieties of information that can be obtained from this personnel mechanism. It has been a simple task, for example, to determine the frequency of professional activity in a given specialty and comparatively little time has been required for tabulating information about changes in activity at different periods. Other results that are not mentioned might be treated with equal facility. But, at this stage of the project, it is deemed more important to illustrate the method and to indicate its possibilities than to attempt an exhaustive report of results.

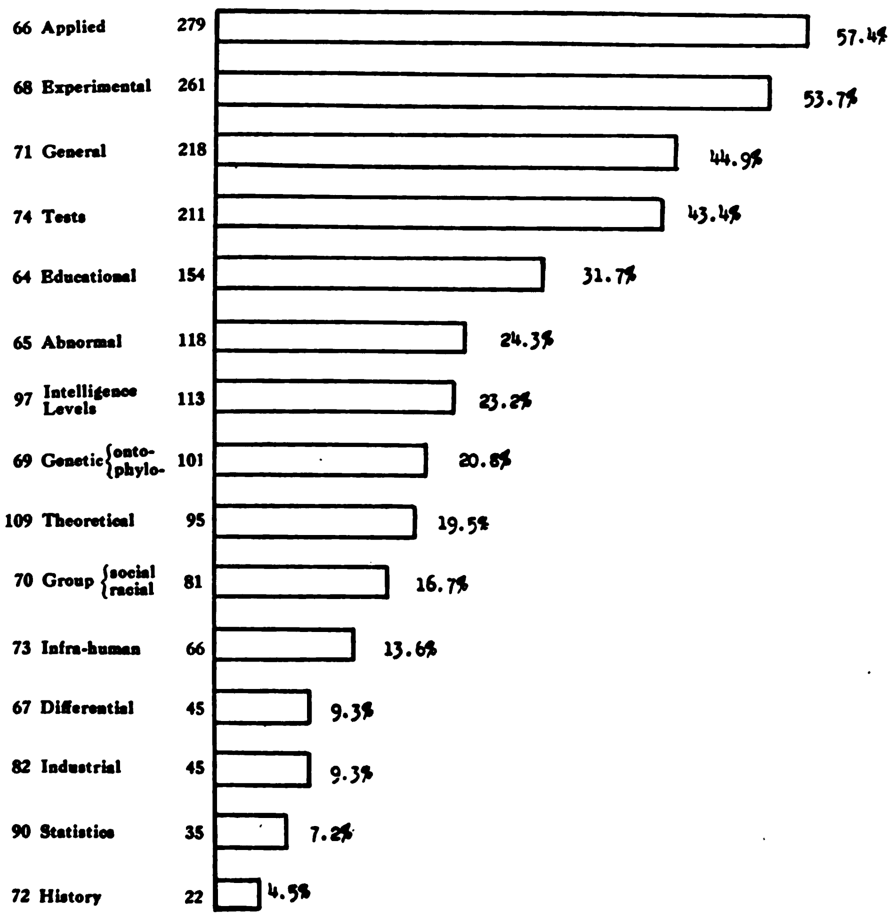

FIGURE 7. Frequency of activity in 15 different subjects as reported by 486 psychologists.

Figure 7 represents the results of fifteen different Findex sorts, each having been made for a single subdivision of psychology. After findexing 486 records, the number reporting activities in a given field or branch[1] was determined by the method described in the preceding pages. The rank order of the different subjects, determined by the number of psychologists who have reported activities therein, is graphically represented. Immediately preceding the name of the special activity appears the code number, and immediately following occurs the number of individuals represented. Horizontal lines indicate the percentage of the entire group occupied in the different specialties. For all of the subjects combined, these 486 psychologists have been recorded 1,844 times. Thus, the average psychologist in America reports activity in slightly less than four (3.8) subjects of specialization, representing a scatter of 25 per cent.[2]

Applied psychology, shown in Figure 7 as the most frequented branch, continues to appear among the subjects of research reported by certain members of the American Psychological Association. Other members of the association report their specialties in more precise terms; yet a study of their personnel records indicates that their activities are often no less "applied" than those which are reported as such. In preparing Findex records, therefore, it has been necessary, for the sake of consistency, to record as applied psychology those activities which publications and reputation justify, even though they were not so designated by the individual himself. Indeed, on account of variation in the use of terminology by different psychologists, this liberty has been necessarily taken with many records.

Figure 8 represents graphically a partial distribution of activities of 279 men and women classified primarily in applied psychology. The figure indicates, for example, that 65.5 per cent are also classified in experimental psychology, 61.2 per cent have reported work with mental tests, and more than one-half of the group (51.6 per cent) are classified in educational psychology. More than 47 per cent, illustrated by the horizontal line through the first two columns, are credited both with experimental psychology and with mental tests. Nearly 29 per cent, it will be observed, are classified also in each of the first three groups. Near the base line of the figure where four

[1]The term "branch" is used in this report to indicate subdivisions or specialties of psychology, which is broadly called a "field" of activity co-ordinate with other fields representing the arts, sciences, and technologies. In general, "branch" is used to designate subdivisions of psychology which are characterized by distinctive materials. "Subject" is used more loosely, as in the Year Book of the American Psychological Association, to include fields, branches, and such other subdivisions as methods and points of view.

[2]See also page 29.

or more subjects in combination with applied psychology are represented, it is shown that the percentage of overlapping becomes comparatively low. Dotted lines are inserted in two instances merely to prevent a confluence of solid lines. The point of termination on the outer border of the column representing genetic psychology indicates the percentage level of the combination of subjects represented by the dotted lines.

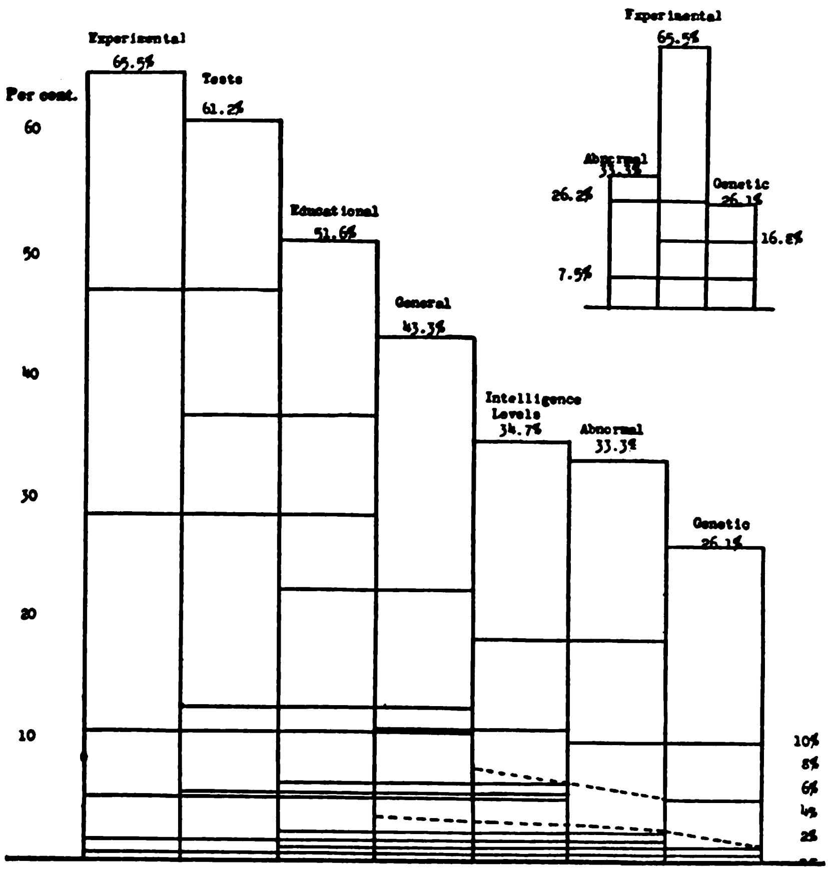

FIGURE 8. Distribution of activities of 279 men and women in applied psychology.

The small inset in the upper right-hand corner of Figure 8 represents the relation of two branches less frequented by applied psychologists, abnormal and genetic, to the most frequented specialty, experimental psychology. The horizontal lines indicate that 26.2 per cent are classified in abnormal and experimental, 16.8 per cent in experimental and genetic psychology, and 7.5 per cent in all three subjects.

TABLE 1.—Extension of activities into related fields as reported by 486 psychologists—Number distribution

Code Number	Psychological Specialty	Total	Related Fields													
			Education (78)	Philosophy (86)	Physiology (87)	Sociology (89)	Religion (88)	Industry (82)	Hygiene (81)	Statistics (90)	Esthetics (79)	Biology (76)	Business (77)	Medicine (85)	Anthropology (75)	Law (83)
66	Applied	279	193	91	50	47	22	40	34	28	17	19	25	27	13	3
68	Experimental	261	159	104	62	32	13	32	22	27	19	25	23	18	14	2
71	General	218	114	104	58	21	16	20	13	11	17	19	10	8	8	2
74	Tests	211	137	69	34	34	10	28	21	30	11	13	17	16	13	2
64	Educational	153	153	43	18	19	8	14	15	17	8	6	9	5	8	0
65	Abnormal	118	65	35	34	25	6	13	24	8	5	11	6	25	7	2
97	Intelligence Levels	113	82	30	22	18	0	11	17	18	4	7	9	18	6	1
69	Genetic	101	71	33	26	17	9	8	15	7	5	13	3	7	11	1
109	Theoretical	95	36	70	19	16	20	4	6	6	9	9	2	1	6	0
70	Group	81	49	34	17	30	6	7	13	11	4	13	4	5	7	1
73	Infra-human	66	36	27	18	9	5	9	7	3	3	13	4	5	4	0
67	Differential	45	28	14	7	9	5	8	5	3	2	5	1	9	3	0
82	Industrial	45	22	12	9	7	2	45	2	9	2	3	17	1	0	0
90	Statistics	35	23	4	6	10	0	9	2	35	0	2	7	0	3	0
72	History	22	8	17	2	4	2	0	0	0	3	1	0	0	2	0

Table 1 represents the number of psychologists in each of fifteen specialties who are working in fourteen related fields. The term "related" has been used to designate those fields of activity, not primarily psychological, but in which at least a portion of the work of psychologists is conducted. In some instances, of course, the psychological work is conducted in related fields as a parallel activity, but more frequently it represents an interaction of projects or interests.

In Table 2 the same distribution among psychological subjects and related fields is presented in percentages. The fourteen related fields are arranged in the order of frequency of representation therein.

TABLE 2.—Extension of activities into related fields as reported by 486 psychologists—Percentage distribution

Code Number	Psychological Specialty	Total	Related Fields													
			Education (78)	Philosophy (86)	Physiology (87)	Sociology (89)	Religion (88)	Industry (82)	Hygiene (81)	Statistics (90)	Esthetics (79)	Biology (76)	Business (77)	Medicine (85)	Anthropology (75)	Law (83)
66	Applied	279	69.2	32.6	17.9	16.8	7.9	14.3	12.2	10.0	6.1	6.8	9.0	9.7	4.7	1.1
68	Experimental	261	60.9	39.8	23.7	12.3	5.0	12.3	8.4	10.3	7.3	9.6	8.8	6.9	5.4	.8
71	General	218	52.3	47.7	26.6	9.6	7.3	9.2	6.0	5.0	7.8	8.7	4.6	3.7	3.7	.9
74	Tests	211	64.9	32.7	16.1	16.1	4.7	13.3	10.0	14.2	5.2	6.2	8.1	7.6	6.2	.9
64	Educational	153	100.0	28.1	11.8	12.4	5.2	9.2	9.8	11.1	5.2	3.9	5.9	3.3	5.2	0
65	Abnormal	118	55.1	29.7	28.8	21.2	5.1	11.0	20.3	6.8	4.2	9.3	5.1	21.2	5.9	1.7
97	Intelligence Levels	113	72.6	26.6	19.5	15.9	0	9.7	15.0	15.9	3.5	6.2	8.0	15.9	5.3	.9
69	Genetic	101	70.3	32.7	25.7	16.8	8.9	7.9	14.9	6.9	5.0	12.9	3.0	6.9	10.9	1.0
109	Theoretical	95	37.9	73.7	20.0	16.8	21.1	4.2	6.3	6.3	9.5	9.5	2.1	1.1	6.3	0
70	Group	81	60.5	42.0	21.0	37.0	7.4	8.6	16.0	13.6	4.9	16.0	4.9	6.2	8.6	1.2
73	Infra-human	66	54.5	40.9	27.3	13.6	7.6	13.6	10.6	4.5	4.5	19.7	6.1	7.6	6.1	0
67	Differential	45	62.2	31.1	15.6	20.0	11.1	17.8	11.1	6.7	4.4	11.1	2.2	20.0	6.7	0
82	Industrial	45	48.9	26.7	20.0	15.6	4.4	100.0	4.4	20.0	4.4	6.7	37.8	2.2	0	0
90	Statistics	35	65.7	11.4	17.1	28.6	0	25.7	5.7	100.0	0	5.7	20.0	0	8.6	0
72	History	22	36.4	77.3	9.1	18.2	9.1	0	0	0	13.6	4.5	0	0	9.1	0

In parentheses with each of the related fields, as in Table 1, there appears the corresponding code number. A striking feature in these tables about which there may be some doubt is the representation of educational psychologists in the related field of education. It is possible that there are investigators working on problems in educational psychology who may not be properly classed in the field of education; but the reports on which this analysis is based emphasize the infrequency of this occurrence. If these representatives of educational psychology have not reported professional employment in education, they have at least indicated preparation in the field by reporting, for example, education as a minor subject for the doctorate. According to the count of Findex records, no report has been made by an individual who seemed to merit classification in educational psychology without a corresponding classification in education. As the results turn out there is indication that educational psychology is a useless category in the code, except as a subdivision of the related field of education.

Although 100 per cent of educational psychologists have been classified also in education, it appears that 262 individuals have reported work in the field of education. In other words, 58.4 per cent of the number reporting work in education have been classified also in educational psychology. These results have been carefully checked by inspection of each original record in order to test in a systematic manner one of the fundamental features of the Findex procedure.

When the psychology code was originally prepared, it was expected that activities would be reported in education which do not belong to educational psychology. But it was assumed that the combination of position 66 (applied psychology) with position 78 (education) would cover for practical purposes the branch of educational psychology. No position, therefore, was assigned in the early form of the code to a classification of educational psychology.

Not only because there was a desire to re-check the status of educational psychology in relation to education, but also because there was need for a check on the reliability of sorting by means of combined positions for subjects unspecified in the code, a new classification, educational psychology, was later assigned to position 64. Each original record was then re-inspected and appropriate Findex records were made for the added classification.

The later procedure demonstrated that no records of educational psychologists were buried under the original plan. Although position 66 (applied psychology) combined with position 78 (education) produces 40 names not produced by position 64 (educational psychology), all of the 153 names of educational psychologists may be obtained by either procedure. By use of the combination of positions (66+78) there is required, in addition to the mechanical selection, a hand sort of the 193 records produced in order to eliminate the 40 names which represent both applied psychology and education but do not represent educational psychology.

By similar combinations of items in the present code, it is always possible to get at least a preliminary selection of persons representing a specialty which may not appear as such in the code. If the Findex

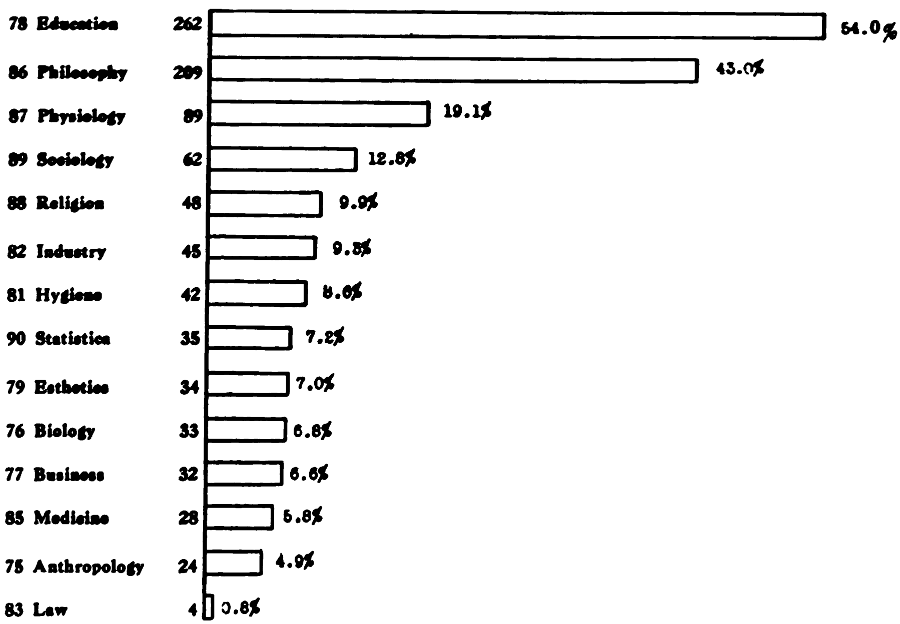

FIGURE 9. Frequency of activity in 14 related fields as reported by 486 psychologists.

operator, for example, were asked to furnish the names of psychologists who might be able to advise on a laboratory problem pertaining to musical abilities, position 79 (esthetics) and 68 (experimental) would be rodded. From the names thus obtained—less than 10 per cent of the total—the desired specialists would be further selected by personal inspection of original records.

Figure 9 shows graphically and comparatively the frequency with which psychologists report activities in the related fields which appear in Table 1. The method of illustration is the same as that followed in Figure 7. Perhaps it is not surprising to note that the frequency of representation in education and philosophy combined (471) is almost equal to the combined representation in the remaining 12 related fields (476). In all of these related fields there is representation 947 times by 486 psychologists indicating an average distribution of activities between two related fields. The scatter of activities, therefore, is not completely described by consideration of the subjects of psychology alone. In addition to the average of 3.8 subjects of specialization among which activities are scattered (cf. page 23) there should also be added the average of 2 occurring in these related fields.

The average American psychologist is thus distributing his activities among nearly six different subjects, if we permit such gross misrepresentations as applied and experimental psychology to remain in the list. When the total of 29 fields and branches is considered, however, the scatter is reduced to 20 per cent. On the other hand, we should not fail to recognize that representation in these subjects has been recorded only where evidence has been found. It is eminently safe to conclude that some, if not many, instances have been omitted either by the psychologists in preparing their original records or by the author in recording the reported facts.

The American psychologist should not be condemned on this evidence for lack of specialization. Several facts may be noted which greatly improve the appearance of his scientific status. In the first place the impropriety of treating applied psychology as a distinctive branch has been observed. The same discrepancy may also be cited in connection with experimental psychology, theoretical psychology, statistics, and so forth. Such activities often are treated merely, and doubtless more properly, as recognized methods of working in certain branches which are fairly distinctive because they represent reasonably definite materials. Is it not equally absurd, for example, to speak of experimental psychology and experimental physics? In the one, experimental is used as though it separated sheep from goats; in the other, it seems to be taken for granted.

Lack of precision in defining subjects of psychology is clearly responsible for some of the scatter which these results indicate. No doubt, the high overlap, as shown in Tables 3 and 4, between statistics and tests may thus be explained. Other instances of high overlap doubtless may be similarly explained. If this is a correct explanation, it appears that the overlap which has been pointed out is more to be praised than blamed for the value of specialists in a single branch which is defined by the nature of the subject-matter may be greatly enhanced by familiarity with several methods.

TABLE 3.—Scatter of activities as reported by 486 men and women for different subjects of psychology—Number distribution

Code Number	Psychological Specialty	Total	Psychological Specialty: Applied	Experimental	General	Tests	Educational	Abnormal	Intelligence Levels	Genetic	Theoretical	Group	Infra-human	Differential	Industrial	Statistics	History
66	Applied	279	279	183	121	171	144	93	97	73	33	53	33	41	40	28	5
68	Experimental	261	183	261	167	162	104	87	90	65	36	52	62	36	32	27	8
71	General	218	121	167	218	96	64	54	42	42	48	33	48	16	20	11	13
74	Tests	211	171	162	96	211	102	76	113	51	22	54	36	32	28	30	0
64	Educational	153	144	104	64	102	153	42	65	47	13	28	19	22	14	17	3
65	Abnormal	118	93	87	54	76	42	118	55	34	14	26	20	25	13	8	1
97	Intelligence levels	113	97	90	42	113	65	55	113	34	10	31	20	19	11	18	1
69	Genetic	101	73	65	42	51	47	34	34	101	16	33	23	20	8	7	4
109	Theoretical	95	33	36	48	22	13	14	10	16	95	20	8	3	4	6	12
70	Group (Social and Racial)	81	53	52	33	54	28	26	31	33	20	81	14	8	7	11	5
73	Infra-human	66	33	62	48	36	19	20	20	23	8	14	66	6	9	3	3
67	Differential	45	41	33	16	32	22	25	19	20	3	8	6	45	8	3	0
82	Industrial	45	39	32	20	28	14	13	9	8	4	7	9	8	45	9	0
90	Statistics	35	28	27	11	30	16	8	18	7	6	11	3	2	9	35	0
72	History	22	5	8	13	0	3	1	1	4	12	5	3	0	0	0	22
Total in each specialty			279	261	218	211	153	118	113	101	95	81	66	45	45	35	22
Percentage in each specialty			57.4	53.8	44.9	43.4	31.5	24.3	23.3	20.8	19.5	16.7	13.6	9.3	9.3	7.2	4.5

Another division, general psychology, about which there is additional uncertainty, is defined in the working code as the psychology of the normal, adult, human individual. This brief statement, like some other definitions in the code, fails to cover fully all of the considerations. At first thought it may seem that all psychologists who have recognized qualifications for independent research must have basic preparation in general psychology. Instead of slightly less than 45 per cent representation in general psychology, should there not be 100 per cent? Obviously, general psychology is the fundamental activity from which all of the branches spring. On theoretical grounds, then, it seems absurd to treat general psychology as a specialty; but study of original records indicated that it ought to be retained in this project to provide for activities that could not be specifically referred to such branches as abnormal, genetic, collective, or infra-human psychology. Where the activities were definitely conducted in such branches, the classification of general psychology was not deemed pertinent without special reason. When the records of psychologists appeared who had furnished orientational contributions to their science in the form of text-books, fundamentally trained students, or professional leadership, they were credited with general psychology, even though their published research contributions placed them in one or more of the special branches. Young psychologists, whose records evidenced no inclination to concentrate in special branches or evidenced an inclination to avoid rigid adherence to limited tendencies in the science, were also given this classification.

One feature which probably deserves special note is the overlap of activities between philosophy and the different divisions of psychology. Of those who have been classified in theoretical psychology, as indicated in Table 2, 73.7 per cent are also classified in philosophy. Similarly, it is found that 77.3 per cent of those in the history of psychology are also in philosophy. It is significant that the average overlap for the remaining subjects of psychology and philosophy is 34.8 per cent; and less than 50 per cent of the psychologists working in any of these fields report work in philosophy.

The significance of this sharp split in the overlap with philosophy is emphasized when a corresponding split in Table 4 is observed. As indicated in that table, 37.9 per cent of the people who have been classified in theoretical psychology are also classified in experimental psychology. Similarly, it is found that 36.4 per cent of the people in the history of psychology are also in experimental psychology. The average overlap between philosophy and the remaining divisions of psychology is 76.9 per cent with the lowest overlap represented by 64.2 per cent.

TABLE 4.—Scatter of activities as reported by 486 men and women for different specialties in psychology—Percentage distribution

Code Number	Psychological Specialty	Total	Psychological Specialty: Applied	Experimental	General	Tests	Educational	Abnormal	Intelligence Levels	Genetic	Theoretical	Group	Infra-human	Differential	Industrial	Statistics	History
66	Applied	279	100.0	65.6	43.4	61.2	51.6	33.3	34.8	26.2	11.8	19.0	11.8	14.7	14.3	10.0	1.8
68	Experimental	261	70.1	100.0	64.0	62.0	39.8	33.3	34.5	24.9	13.8	19.9	23.7	13.8	12.3	10.3	3.1
71	General	218	55.5	76.6	100.0	44.0	29.4	24.8	19.3	19.3	22.0	15.1	22.0	7.3	9.2	5.0	5.9
74	Tests	211	81.0	76.8	45.5	100.0	48.3	36.0	53.6	24.2	10.4	25.6	17.1	15.2	13.3	14.2	0
64	Educational	153	94.1	68.0	41.8	66.7	100.0	27.5	42.5	30.8	8.5	18.3	12.4	14.4	9.2	11.1	2.0
65	Abnormal	118	78.8	73.7	45.8	64.4	35.6	100.0	46.6	28.8	11.9	22.0	16.9	21.2	11.0	6.8	.8
97	Intelligence Levels	113	85.8	79.6	37.1	100.0	57.6	48.7	100.0	30.1	8.9	27.4	17.7	16.8	9.7	15.9	.9
69	Genetic	101	72.3	64.4	41.6	50.5	46.5	33.7	33.7	100.0	15.8	32.7	22.8	19.8	7.9	6.9	4.0
109	Theoretical	95	34.7	37.9	50.5	23.2	13.7	14.7	10.5	16.8	100.0	21.1	8.4	3.2	4.2	6.3	12.6
70	Group	81	65.4	64.2	40.7	66.7	34.6	32.1	38.3	40.7	24.7	100.0	17.3	9.9	8.6	13.6	6.2
73	Infra-human	66	50.0	93.9	72.7	54.5	28.8	30.3	30.3	34.8	12.1	21.2	100.0	9.1	13.6	4.5	4.5
67	Differential	45	91.1	80.0	35.6	71.1	48.9	55.6	42.2	44.4	6.7	17.8	13.3	100.0	17.8	6.7	0
82	Industrial	45	86.7	71.1	44.4	62.2	31.1	28.9	20.0	17.8	8.9	15.6	20.0	17.8	100.0	20.0	0
90	Statistics	35	80.0	77.2	31.4	85.7	45.7	22.9	51.4	20.0	17.1	31.4	8.6	5.7	25.7	100.0	0
72	History	22	22.7	36.4	59.1	0	13.6	4.5	4.5	18.2	54.5	22.7	18.6	0	0	0	100.0

The difference between the combination of activities with philosophy on the one hand and with experimental psychology on the other is illustrated in Figure 10. The curves indicate that psychologists credited with experimental psychology tend to overlap more than the philosophers in the branches which are most frequented. The psychologists credited with philosophy overlap more than the experimentalists in the less frequented subjects. The points of conspicuous divergence, exclusive of the experimental group itself, appear in the subjects designated as applied, tests, theoretical, and

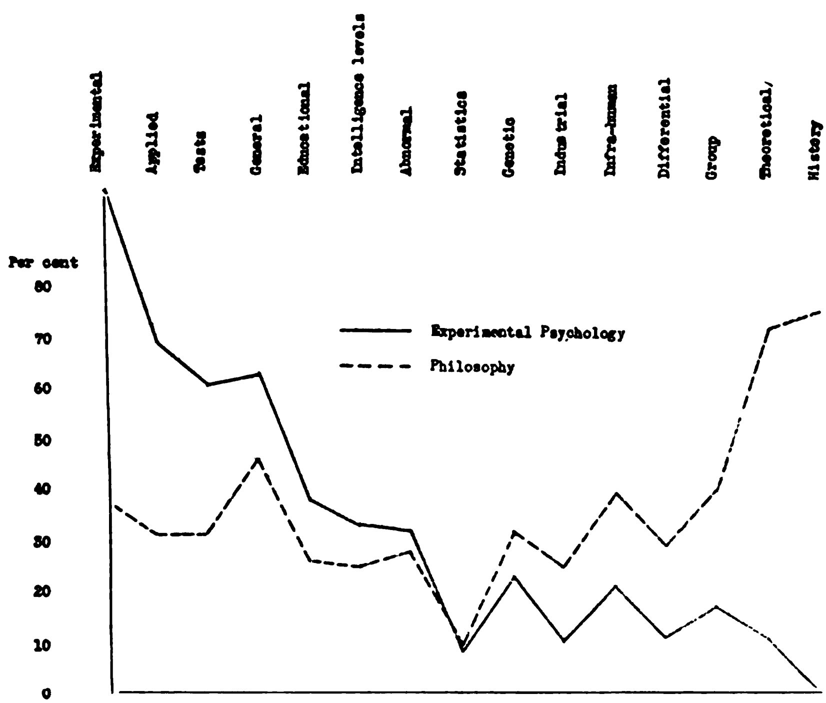

FIGURE 10. Relative frequency in different specialties of psychologists who are experimentalists and who are philosophers.

history. The points of close approximation appear in the branches called intelligence levels, abnormal psychology, statistics, and genetic psychology. If the differences in frequency of representation in the subjects preceding the intersection be interpreted as positive differences, and those following the intersection as negative, it will be observed that the subjects are arranged so that there is a descending order in the differences. In general, the activities in the various branches of psychology with reference to philosophy on the one hand and to experimental psychology on the other are as follows:

where there is a high overlap in philosophy there is a low overlap in experimental psychology; where there is a low overlap in philosophy there is a high overlap in experimental psychology. The term experimental psychology, it would seem, is broadly used at present to differentiate a particular kind of scientific background or technique from philosophy.

TABLE 5.—Relation of birth periods of 486 psychologists to subjects in which activity is reported—Number distribution

Code Number	Subject of Activity	Total	Period of birth							
			1860 or earlier	1861–1870	1871–1875	1876–1880	1881–1885	1886–1890	1891–1895	1896 or later
66	Applied..........	270	10	42	41	47	45	58	26	1
68	Experimental....	261	8	38	43	47	44	49	30	2
71	General..........	218	12	44	46	33	36	31	15	1
74	Tests............	208	1	26	25	34	44	51	27	0
64	Educational......	152	4	21	29	31	29	27	11	0
65	Abnormal........	117	3	16	23	19	23	23	9	1
97	Intelligence Levels.........	110	1	12	15	21	20	30	11	0
69	Genetic..........	101	9	25	19	23	10	12	3	0
109	Theoretical......	95	9	30	19	12	8	13	4	0
70	Group...........	81	5	15	15	11	12	12	11	0
73	Infra-human....	65	2	11	13	16	5	17	1	0
67	Differential......	43	0	5	6	11	8	10	2	1
82	Industrial.......	41	0	1	4	8	9	12	7	0
72	History..........	21	2	9	4	2	1	2	1	0
	Total.........	1783	66	295	302	315	294	347	158	6

Another fact which further reduces the significance of this scatter of activities among American psychologists is the policy of recording an activity in every instance where it could be justified. Where there was doubt about the propriety of recording or omitting a classification, the rule always was to make the entry. Special training, professional reputation, a publication, or an announced investigation provided acceptable reasons for recording an activity in a given classification. This procedure was important because research people having several qualifications are often sought. The expediency naturally produces numbers for some of the branches which will be questioned by eminent specialists, but objections on the ground of inflated numbers can hardly be valid for comparative treatment of the results. When, for instance, there is need to consider possible candidates for

TABLE 6.—Relation of birth periods of 486 psychologists to subjects in which activity is reported—Percentage distribution

Code Number	Subject of Activity	Total	Period of birth							
			1860 or earlier	1861–1870	1871–1875	1876–1880	1881–1885	1886–1890	1891–1895	1896 or later
66	Applied.........	270	3.7	15.6	15.2	17.4	16.7	21.5	9.6	.4
68	Experimental....	261	3.1	14.6	16.5	18.0	16.9	18.8	11.5	.8
71	General.........	218	5.5	20.2	21.1	15.1	16.5	14.2	6.9	.5
74	Tests...........	208	.5	12.5	12.0	16.3	21.2	24.5	13.0	0
64	Educational.....	152	2.6	13.8	19.1	20.4	19.1	17.8	7.2	0
65	Abnormal.......	117	2.6	13.7	19.7	16.2	19.7	19.7	7.7	.9
97	Intelligence Levels........	110	.9	10.9	13.6	19.1	18.2	27.3	10.0	0
69	Genetic.........	101	8.9	24.8	18.8	22.8	9.9	11.9	3.0	0
109	Theoretical......	95	9.5	31.6	20.0	12.6	8.4	13.7	4.2	0
70	Group..........	81	6.2	18.5	18.5	13.6	14.8	14.8	13.6	0
73	Infra-human....	65	3.1	16.9	20.0	24.6	7.7	26.2	1.5	0
67	Differential......	43	0	11.6	14.0	25.6	18.6	23.3	4.7	2.3
82	Industrial.......	41	0	2.4	9.8	19.5	22.0	29.3	17.1	0
72	History.........	21	9.5	42.9	19.0	9.5	4.8	9.5	4.8	0
	Total.........	1783	3.7	16.5	16.9	17.7	16.5	19.5	8.9	0.3

a given position, it is highly desirable that the mechanical sort shall produce the names of all individuals, whose qualifications approximate the specifications. Final selection, obviously, should be made by personal inspection of records after the preliminary selection has been mechanically accomplished.

Inclusion of psychologists of all ages is a further explanation of the wide scatter of activities. Presumably, many older psychologists have been active in more subjects than the younger members of the group can possibly represent. In the early history of modern psychology the specialties of today were almost nothing. The older school helped to develop embryonic tendencies which have become specialties, and in so doing, the activities of the older school became rather widely distributed. Moreover, the older representatives have

TABLE 7.—Relation of birth periods of 486 psychologists to activities reported in related fields—Number distribution

Code Number	Field of Activity	Total	Period of birth							
			1860 or earlier	1861–1870	1871–1875	1876–1880	1881–1885	1886–1890	1891–1895	1896 or later
78	Education.......	262	14	46	42	45	42	42	25	1
86	Philosophy......	209	16	56	43	36	24	20	12	1
87	Physiology......	89	6	11	14	17	16	15	9	0
89	Sociology........	62	5	12	7	7	10	16	5	0
88	Religion.........	48	6	17	10	7	3	1	1	0
82	Industry........	45	0	2	4	8	9	12	7	0
81	Hygiene.........	42	2	12	9	8	5	3	2	1
90	Statistics........	35	1	1	3	5	3	15	6	0
79	Esthetics........	34	5	9	6	7	2	5	0	0
76	Biology.........	33	1	6	10	6	2	3	3	0
77	Business........	32	1	1	2	3	4	11	8	0
85	Medicine........	28	0	5	5	7	2	7	1	0
75	Anthropology....	24	3	8	8	3	0	1	1	0
83	Law............	4	0	2	1	0	0	1	0	0
	Total.........	947	60	187	164	159	122	152	80	3

had more time for expansion of activities than have the younger members of the group. And when additional subdivisions have appeared in the field, the more experienced psychologists have probably kept apace with developments by sharing, at least during early stages, the new activities with the later and more highly developed specialists.

To secure objective data on these possibilities, Tables 5-8 have been prepared. These tables furnish number and percentage distributions for the subjects of psychology and the related fields grouped according to periods of birth of the representatives. The totals in some instances are different than those given in other tables because the date of birth was not available for all individuals. Eight successive birth periods beginning with "1860 or earlier" and ending with "1896 or later" are considered. Because the representatives born before 1870 are relatively few, two five-year intervals have been lumped in one interval for 1861–1870. Thereafter the groups represent five-year intervals. There are so few psychologists whose births occurred in or before 1860 and in or after 1896 that those two groups are practically negligible. We may be quite certain, too, that the number of records for the birth group of 1891–1895 has not yet reached the maximum. The decline in numbers after 1890 does not mean a smaller number now entering psychology, but merely incomplete returns from many who will eventually qualify in the field.

TABLE 8.—Relation of birth periods of 486 psychologists to activities reported in related fields—Percentage distribution

Code Number	Field of Activity	Total	Period of birth							
			1860 or earlier	1861–1870	1871–1875	1876–1880	1881–1885	1886–1890	1891–1895	1896 or later
78	Education	262	5.3	17.6	16.0	17.2	16.0	16.0	9.5	.4
86	Philosophy	209	7.7	26.3	20.6	17.2	11.5	9.6	5.7	.5
87	Physiology	89	6.7	12.4	15.7	19.1	18.0	16.9	10.1	0
89	Sociology	62	8.1	19.4	11.3	11.3	16.1	25.8	8.1	0
88	Religion	48	12.5	35.4	20.8	14.6	6.3	2.1	2.1	0
82	Industry	45	0	4.4	8.9	17.8	20.0	26.7	15.6	0
81	Hygiene	42	4.8	28.6	21.4	19.0	11.9	7.1	4.8	2.4
90	Statistics	35	2.9	2.9	8.6	14.3	8.6	42.9	17.1	0
79	Esthetics	34	14.7	26.5	17.6	20.6	5.9	14.7	0	0
76	Biology	33	3.0	18.2	30.3	18.2	6.1	9.1	9.1	0
77	Business	32	3.1	3.1	6.3	9.4	12.5	34.4	25.0	0
85	Medicine	28	0	17.9	17.9	25.0	7.1	25.0	3.6	0
75	Anthropology	24	12.5	33.3	33.3	12.5	0	4.2	4.2	0
83	Law	4	0	50.0	25.0	0	0	25.0	0	0

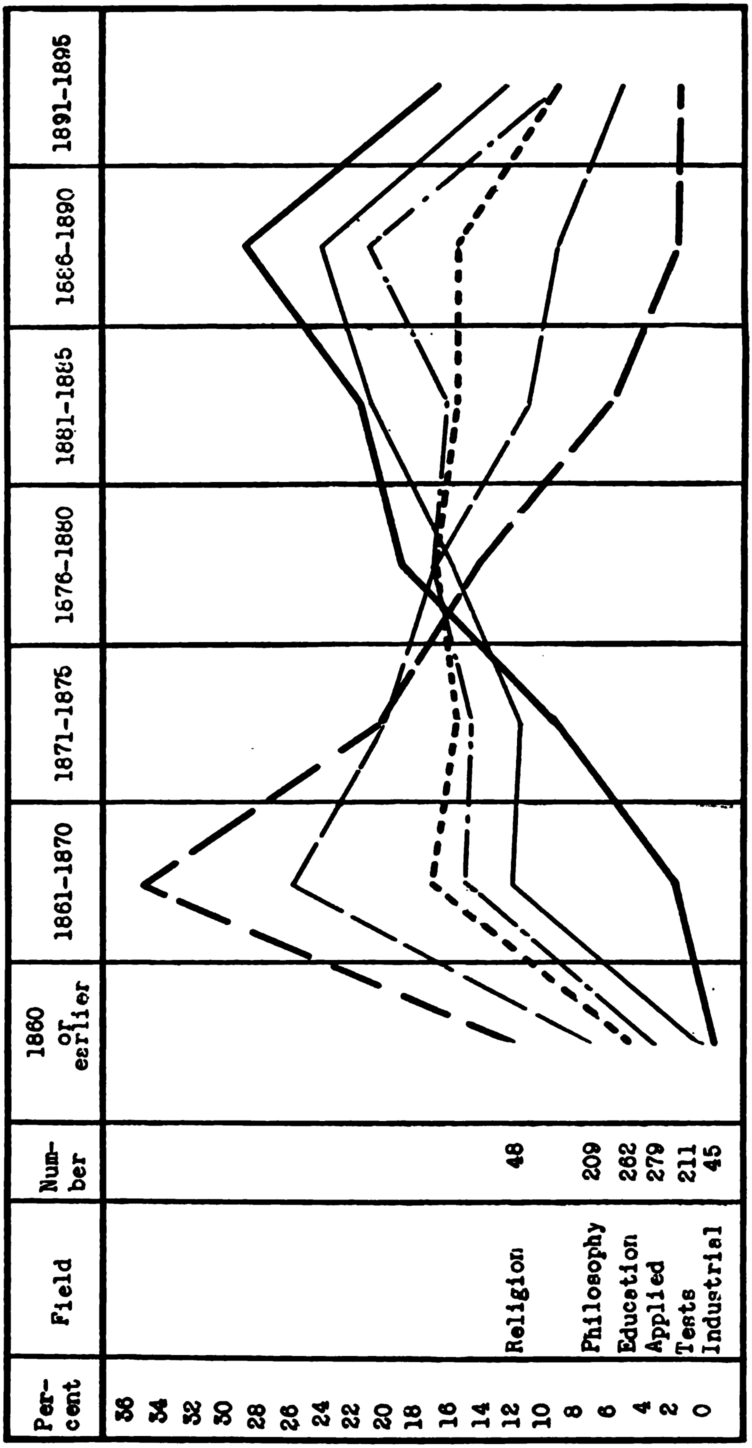

FIGURE 11. Frequency in six activities as reported by psychologists of different birth periods.

In order to show an outstanding change in emphasis by the younger psychologists, Figure 11, based on selections from Tables 5–8, is presented. Three fields of activity—religion, philosophy, and education—provided opportunity for some of the earlier developments in psychological inquiry. The three subjects of psychology which have been selected—applied psychology, tests, and industrial psychology—may be numbered among later developments. Furthermore, religion and philosophy represent a speculative tendency; the three later specialties are representative of a practical interest. As shown in Figure 11, the psychologists most frequently reporting specialties in religion and philosophy belong to the earlier age groups; their numbers are relatively low in the practical specialties. In contrast, the representatives in these practical activities are more numerous in the later groups while the representatives of the theoretical activities are less numerous. The crossing of these two tendencies occurs conspicuously in the age group of 1876–1880.

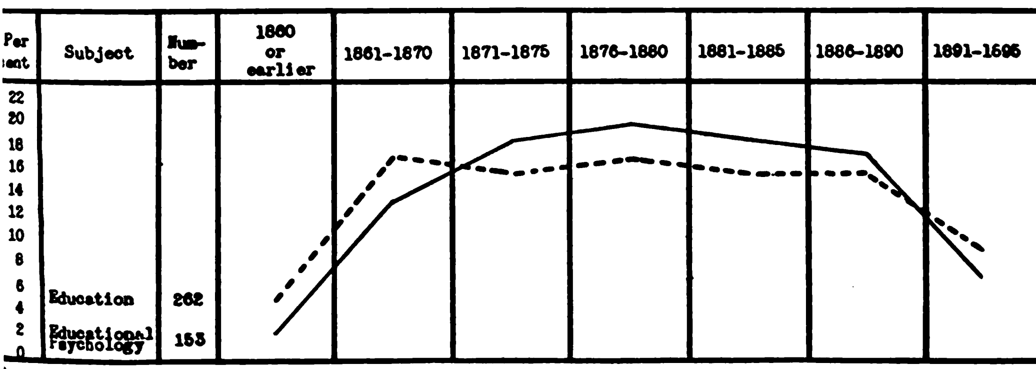

FIGURE 12. Curves showing the representation in education and educational psychology of psychologists who belong to different age groups.
Broken line, education. Solid lines, educational psychology.

The drop of the curves in the last column is most certainly due to incomplete returns from the group which is just attaining recognition. When this group has been fully recognized it seems probable that the trend of the curves taken after 1876-1880 will be continued in the interval 1891–1895.

Especially noteworthy is the curve representing the trend in education. This curve, it will be observed, is for the 262 psychologists in education, not the 153 educational psychologists. Throughout the entire series of age groups the curve for education tends to run in an intermediate course. Only in the age group of 1876–1880, where the curves cross, does education fail to maintain in all respects this intermediate relation. The curve for tests is the only one which does not reverse its relative position above or below education in the age group of 1876–1880.

In Figure 12, curves have been plotted showing, comparatively, the frequency of representation in education and educational psychology. The curve for education is the same as that appearing in Figure 11. The two curves cross in the age group 1871–1875, one period earlier than any of the other trends which have been illustrated. After this intersection, educational psychology tends to run on a fairly level plane, corresponding closely to education except that it is about four per cent higher.

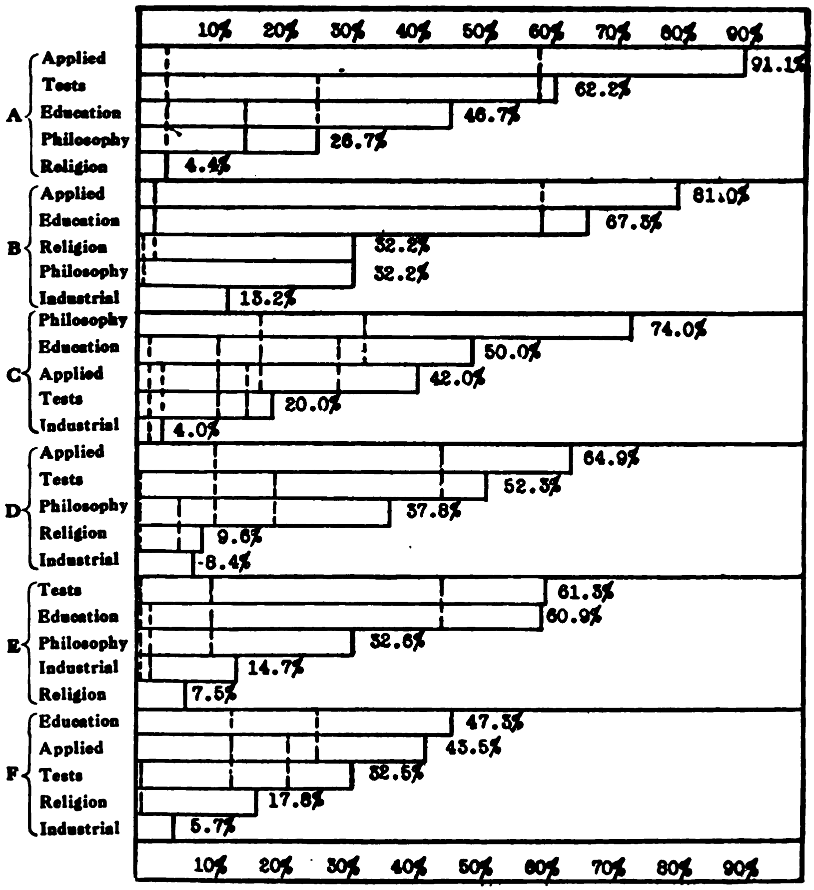

FIGURE 13

A. 45 Psychologists reporting work in Industrial Psychology.
B. 211 Psychologists reporting work in Tests.
C. 50 Psychologists reporting work in Religion.
D. 262 Psychologists reporting work in Education.
E. 279 Psychologists reporting work in Applied Psychology.
F. 209 Psychologists reporting work in Philosophy.

TABLE 9.—Number and percentage distributions of 486 men and women psychologists in various kinds of activity

	Kind of Activity	Total Representation	415 Men			71 Women		
			Number	Percent of Total Representation	Percent of 415	Number	Percent of Total Representation	Percent of 71
Psychological Specialties	Abnormal	118	90	76.3	21.7	28	23.7	39.4
	Applied	279	235	84.2	56.6	44	15.8	62.0
	Differential	45	34	75.6	8.2	11	24.4	15.5
	Educational	154	129	83.8	31.1	25	16.2	35.2
	Experimental	261	213	81.6	51.3	48	18.4	67.6
	General	218	184	84.4	44.3	34	15.6	47.9
	Genetic	101	82	81.1	19.8	19	18.9	26.8
	Group	81	67	82.7	16.1	14	17.3	19.7
	Historical	22	22	100.0	5.3	0	0	0
	Industrial	45	44	97.8	10.6	1	2.2	1.4
	Infra-human	66	58	87.9	14.0	8	12.1	11.3
	Mental Tests	211	168	79.6	40.5	43	20.4	60.6
	Theoretical	95	89	93.7	21.4	6	6.3	8.5
	Intelligence Levels	113	89	78.7	21.4	24	21.3	33.8
Related Fields	Anthropology	24	24	100.0	5.8	0	0	0
	Biology	33	30	90.9	7.2	3	9.1	4.2
	Business	32	28	87.5	6.7	4	12.5	5.6
	Education	262	225	85.9	54.2	37	14.1	52.1
	Esthetics	34	23	67.9	5.5	11	32.1	15.5
	Ethics	63	58	92.1	14.0	5	7.9	7.0
	Hygiene	42	35	83.3	8.4	7	16.7	10.0
	Law	4	4	100.0	1.0	0	0	0
	Logic	27	25	92.6	6.0	2	7.4	2.8
	Medicine	28	24	85.7	5.8	4	14.3	5.6
	Philosophy	209	183	87.6	44.1	26	12.4	36.6
	Physiology	89	73	82.0	17.6	16	18.0	22.5
	Religion	48	45	93.7	10.8	3	6.3	4.2
	Sociology	62	51	82.3	12.3	11	17.7	15.5
	Statistics	35	34	97.1	8.2	1	2.9	2.2

In Figure 13 there appears more detailed analysis of the six special activities which have been illustrated in Figure 11. In each kind of activity percentages of psychologists, who have been credited also with the remaining five activities, are illustrated by horizontal lines followed by the percentage which is represented. The legend with the table gives the total number in each field or subject which is thus analyzed. By referring to the dotted lines which run vertically through two or more of the columns and to the scale of percentages at the top of the figure, the amount of overlap in two or more activities can be readily read.

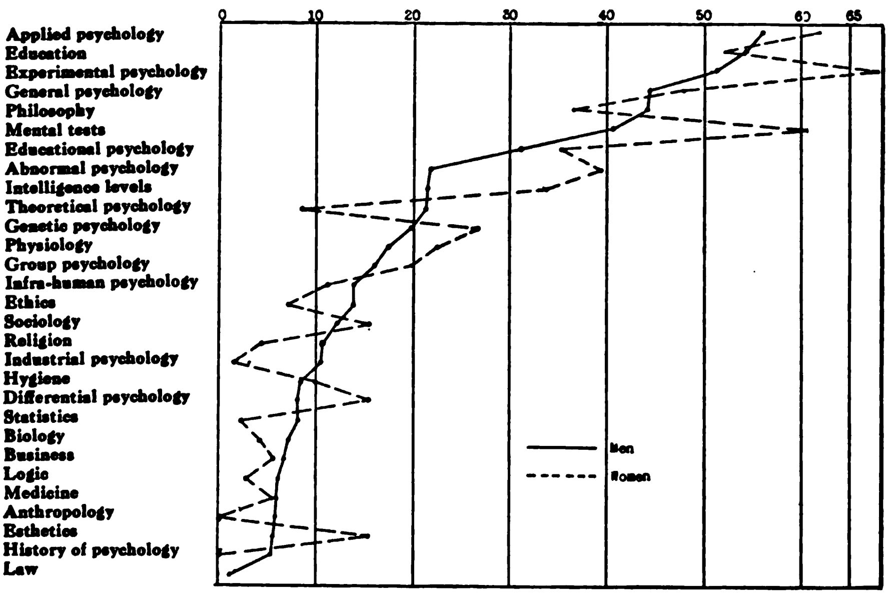

FIGURE 14. Percentage curves showing comparative representation of men and women psychologists in different subjects. The vertical lines and corresponding numbers above represent percentages.

Seventy-one, or 14.6 per cent, of the 486 psychologists considered in this report are women. Table 9 shows how their activities are respectively distributed among branches of psychology and fields related thereto. In Table 10 all of the divisions arranged in rank order, determined by the relative representation of men and women in each, are presented. In three of them—anthropology, history of psychology, and law—it appears that no women are represented. In no activity has the number of women equalled or exceeded the num-

TABLE 10.—Percentage of 486 psychologists who are men and women in various fields and branches arranged in rank order

Rank	Subject or Field	Men	Women
1	Anthropology	100.0	0
2	History of psychology	100.0	0
3	Law	100.0	0
4	Industrial psychology	97.8	2.2
5	Statistics	91.1	2.9
6	Religion	93.7	6.3
7	Theoretical psychology	93.7	6.3
8	Logic	92.6	7.4
9	Ethics	92.1	7.9
10	Biology	90.9	9.1
11	Infra-human psychology	87.9	12.1
12	Philosophy	87.6	12.4
13	Business	87.5	12.5
14	Education	85.9	14.1
15	Medicine	85.7	14.3
16	General psychology	84.4	15.6
17	Applied psychology	84.2	15.8
18	Educational psychology	83.8	16.2
19	Hygiene	83.3	16.7
20	Group psychology	82.7	17.3
21	Sociology	82.3	17.7
22	Physiology	82.0	18.0
23	Experimental psychology	81.6	18.4
24	Genetic psychology	81.1	18.9
25	Mental tests	79.6	20.4
26	Intelligence levels	78.7	21.3
27	Abnormal psychology	76.3	23.7
28	Differential psychology	75.6	24.4
29	Esthetics	67.9	32.1

ber of men. Women make the highest showing in esthetics and in work pertaining to mental capacity and mental deviation.

In Figure 14 there are shown percentage curves for 415 men and 71 women psychologists, indicating the comparative frequency with which each sex reports activities in the various fields and specialties. The plotted curve for men, marked by a continuous line, represents decreasing frequency of representation from applied psychology to law. For direct comparison in any activity, the curve for women is indicated by a dotted line. The activities in which the percentage of representation among women exceeds by 10 per cent or more that among men are experimental psychology, mental tests, abnormal psychology, intelligence levels, and esthetics. The reverse is true only in theoretical psychology although the difference in industrial psychology is nearly 10 per cent. There are nine subjects comprising applied psychology, philosophy, genetic psychology, ethics, religion, differential psychology, statistics, anthropology, and history of psychology in which the differences between percentages of representation fall between 5 per cent and 10 per cent. In the remaining thirteen activities the differences are less than 5 per cent, but in several of these such a difference may be significant because the maximum representation by either sex is comparatively low.

Table 11 shows the numerical distribution, and Table 12 the percentage distribution, of kinds of activity among varieties of institutional experience. The fields are arranged in two groups, as in previous tables, comprising psychological subjects and related fields. University experience is most frequently reported and in every activity it leads the other institutions in numbers. Even in such activities as industrial psychology, experience with industrial concerns is reported by less than 30 per cent, whereas nearly 85 per cent report university experience. Also, experience in colleges is reported more frequently than with industrial concerns; and more than half of the industrial group report government experience, which includes occupation in federal as well as in state positions. The num-
for "government" are swelled, of course, by the psychologists
Seventy-on in the army during The World War.
in this report are he scope of the present paper to draw conclusions or
respectively distributelizations about the status of American psychology.
lated thereto. In Table to present within such limitations all of the
order, determined by the rel mechanism has at hand. To undertake
in each, are presented. In thequire expansion of a preliminary report
psychology, and law—it appears r pretensions. Primarily, the present
no activity has the number of worities of the existing mechanism for
of trends in science or technology.

TABLE 11.—Experience of 486 psychologists as reported for different kinds of activity—Number distribution

Code Number	Kind of Activity	Total	Institutional Experience							
			University	College	Government	Preparatory School	Private Occupation	Grade School	Industrial Concern	Commercial Concern
66	Applied	279	186	118	104	95	13	20	17	10
68	Experimental	261	225	114	76	75	14	15	10	7
71	General	218	190	101	66	58	11	12	8	5
74	Tests	211	162	92	86	64	7	13	8	8
64	Educational	154	108	73	45	66	4	15	6	1
65	Abnormal	118	80	41	47	37	10	4	3	2
97	Intelligence Levels	113	80	44	48	48	4	7	4	2
69	Genetic	101	77	45	24	39	9	9	4	2
109	Theoretical	95	90	38	16	18	4	4	3	3
70	Group	81	62	41	24	21	2	3	1	2
73	Infra-human	66	51	36	27	18	4	4	5	4
67	Differential	45	34	22	12	21	2	1	1	0
82	Industrial	45	38	13	26	10	3	2	10	7
72	History	22	21	8	0	6	1	2	0	1
78	Education	262	200	120	77	91	12	19	8	5
86	Philosophy	209	163	109	43	47	8	4	5	4
87	Physiology	89	70	33	29	22	9	2	5	2
89	Sociology	62	45	24	21	16	3	3	3	2
88	Religion	48	38	27	7	7	4	2	2	1
81	Hygiene	42	24	18	19	16	5	4	3	0
90	Statistics	35	29	9	17	10	1	0	4	3
79	Esthetics	34	25	18	9	6	4	0	1	4
76	Biology	33	27	15	11	6	0	0	0	1
77	Business	32	28	6	15	5	1	2	7	6
85	Medicine	28	17	7	14	7	5	2	0	0
75	Anthropology	24	17	12	7	7	1	1	1	1
83	Law	4	4	1	1	2	1	0	1	0

TABLE 12.—Experience of 486 psychologists as reported for different kinds of activity—Percentage distribution

Code Number	Kind of Activity	Total	Institutional Experience							
			University	College	Government	Preparatory School	Private Occupation	Grade School	Industrial Concern	Commercial Concern
66	Applied.......	279	66.7	42.3	37.3	34.1	4.7	7.2	6.1	3.6
68	Experimental..	261	86.2	43.7	29.1	28.7	5.4	5.7	3.8	2.7
71	General.......	218	87.2	46.3	30.3	26.6	5.0	5.5	3.7	2.3
74	Tests.........	211	76.8	43.6	40.8	30.3	3.3	6.2	3.8	3.8
64	Educational...	154	70.1	47.4	29.2	42.9	2.6	9.7	3.9	.6
65	Abnormal.....	118	67.8	34.7	39.9	31.4	8.5	3.4	2.5	1.7
97	Intelligence Levels......	113	70.8	38.9	42.5	42.5	3.5	6.2	3.5	1.8
69	Genetic.......	101	76.2	44.6	23.8	38.6	8.9	8.9	4.0	2.0
109	Theoretical....	95	94.7	40.0	16.8	18.9	4.2	4.2	3.2	3.2
70	Group........	81	76.5	50.6	29.6	25.9	2.5	3.7	1.1	2.5
73	Infra-human...	66	77.3	54.5	40.9	27.3	6.1	6.1	7.6	6.1
67	Differential....	45	75.5	48.9	26.7	46.7	4.4	2.2	2.2	0
82	Industrial.....	45	84.4	28.9	57.7	22.2	6.7	4.4	22.2	15.6
72	History.......	22	95.4	36.4	0	27.3	4.5	9.1	0	4.5
78	Education.....	262	76.3	45.8	29.4	34.7	4.6	7.3	3.1	1.9
86	Philosophy....	209	78.0	52.1	20.6	22.5	3.8	1.9	2.4	1.9
87	Physiology....	89	78.7	37.1	32.6	24.7	10.1	2.2	5.6	2.2
89	Sociology......	62	72.6	38.7	33.9	25.8	4.8	4.8	4.8	3.2
88	Religion.......	48	79.2	56.2	14.6	14.6	8.3	4.2	4.2	2.1
81	Hygiene.......	42	57.1	42.9	45.2	38.1	11.9	9.5	7.1	0
90	Statistics......	35	82.9	25.7	48.6	28.6	2.9	0	11.4	8.6
79	Esthetics......	34	73.5	52.9	26.5	17.6	11.8	0	2.9	11.8
76	Biology.......	33	81.8	45.5	33.3	18.2	0	0	0	3.0
77	Business.......	32	87.5	18.8	46.9	15.6	3.1	6.3	21.9	18.8
85	Medicine......	28	60.7	25.0	50.0	25.0	17.9	7.1	0	0
75	Anthropology..	24	70.8	50.0	29.2	29.2	4.2	4.2	4.2	4.2
83	Law..........	4	100.0	25.0	25.0	50.0	25.0	0	25.0	0

With examples such as this report contains brought to the attention of administrators, executives, and others who are influential in shaping the courses of the sciences and technologies, it is believed that the resources of the Research Information Service will be increasingly utilized.

Comparison of the code presented on pages 18-21 with the data that have been embodied in this paper will reveal many possibilities of analysis that have been ignored. It is one of the announced purposes of the Service to respond with all of its resources to requests for such information. Where it is feasible, such facts are made available through publication, but in the absence of such facilities the facts are at the command of those who have use for them and who make known their needs. Section 10 is appended as an illustration of such supplementary data as the Service may properly be asked to furnish. This particular list of names was compiled for the use of the American Psychological Association.

10. List of psychologists who have received stars in American Men of Science.[1]

The first three symbols following the name of a psychologist refer to the three editions of American Men of Science. A fourth character is added only for those who are not members of the American Psychological Association. Where stars have been received for work in a field other than psychology, the field is indicated in parentheses.

KEY TO SYMBOLS.

0—not listed.
x—listed without star.
1—star in edition of 1906.
2—star in edition of 1910.
3—star in edition of 1921.
#—not a member of the American Psychological Association.

1. Angell, F(rank) 1-2-3-#.
2. Angell, James R(owland) 1-2-3.
3. Armstrong, A(ndrew) C(ampbell) 1 (Philosophy) -2 (Philosophy) -3 (Philosophy).
4. Baldwin, Bird T(homas) 0-x-3.
5. Baldwin, J(ames) Mark 1-2-3-#.
6. Bentley, Madison x-2-3.
7. Bingham, W(alter) V(an Dyke) 0-x-3.
8. Bolton, Thaddeus L(incoln) 1-2-3.

[1]Assistance in compiling this list of names has been received from Professor Edwin G. Boring, Secretary of the American Psychological Association, and Doctor Dean R. Brimhall, Assistant Editor of the third edition of American Men of Science.

9. Boring, Edwin G(arrigues) 0-0-3.
10. Bryan, William Lowe 1-2-3.
11. Buchner, Edward F(ranklin) 1-2-3.
12. Calkins, Mary Whiton 1-2-3.
13. Cannon, W(alter) B(radford) 1 (Physiology) -2 (Physiology) -3 (Physiology).
14. Carr, H(arvey) 0-x-3.
15. Cattell, J(ames) McKeen 1-2-3.
16. Dana, Charles L(oomis) 1 (nervous and mental diseases) -2 (*ibid.*) -3 (*ibid.*)
17. Dearborn, Walter F(enno) x-x-3.
18. Delabarre, E(dmund) B(urke) 1-2-3.
19. Dewey, John 1 (Philosophy) -2 (Philosophy) -3 (Philosophy).
20. Dodge, Raymond 1-2-3.
21. Donaldson, H(enry) H(erbert) 1 (Neurology) -2 (Neurology) -3 (Neurology).
22. Dunlap, Knight x-x-3.
23. Farrand, Livingston 1 (Anthropology) -2 (Anthropology) -3 (Anthropology).
24. Ferree, C(larence) E(rrol) 0-x-3.
25. Franklin, Mrs. Fabian (Christine Ladd) 1-2-3.
26. Franz, S(hepherd) I(vory) 1 (Physiology) -2 (Physiology) -3 (Physiology).
27. Fullerton, George S(tuart) 1 (Philosophy) -2 (Philosophy) -3 (Philosophy).
28. Gardiner, H(arry) Norman 1 (Philosophy) -2 (Philosophy) -3 (Philosophy).
29. Goddard, Henry H(erbert) 0-x-3.
30. Hall, G(ranville) Stanley 1-2-3.
31. Hollingworth, H(arry) L(evi) 0-x-3.
32. Holmes, S(amuel) J(ackson) 1 (Zoology) -2 (Zoology) -3 (Zoology).
33. Holt, Edwin B(issell) x-2-3-#.
34. Hunter, Walter S(amuel) 0-0-3.
35. Howes, Ethel (Dench) P(uffer) 0-2-3.
36. Jastrow, Joseph 1-2-3.
37. Judd, Charles H(ubbard) 1-2-3.
38. Ladd, George Trumbull 1-2-3 (deceased).
39. Langfeld, Herbert S(idney) 0-0-3.
40. Leuba, James H(enry) 1-2-3.
41. Lindley, Ernest H(iram) 1-2-3.
42. MacDougall, Robert 1-2-3.

43. Marshall, Henry Rutgers 1-2-3.
44. Martin, Lillien J(ane) x-2-3.
45. Meyer, Adolf 1 (Psychiatry) -2 (Psychiatry) -3 (Psychiatry). -3 (Psychiatry).
46. Meyer, Max 1-2-3.
47. Newbold, W(illia)m Romanie 1 (Philosophy) -2 (Philosophy) -3 (Philosophy).
48. Ogden, R(obert) M(orris) x-x-3.
49. Patrick, G(eorge) T(homas) W(hite) 1 (Philosophy) -2 (Philosophy) -3 (Philosophy).
50. Pillsbury, W(alter) B(owers) 1-2-3.
51. Pintner, Rudolf 0-0-3.
52. Prince, Morton x (Medicine) -2 (Medicine) -3 (Medicine).
53. Sanford, Edmund C(lark) 1-2-3.
54. Scott, Walter Dill x-x-3.
55. Scripture, E(dward) W(heeler) 1-2-3-#.
56. Seashore, C(arl) E(mil) 1-2-3.
57. Sidis, Boris x-2-3.
58. Starbuck, Edwin D(iller) x (Education) -2 (Psychology) -3 (Psychology).
59. Starch, Daniel 0-x-3.
60. Starr, M(oses) Allen 1 (Neurology) -2 (Neurology) -3 (Neurology).
61. Stratton, G(eorge) M(alcolm) 1-2-3.
62. Strong, C(harles) A(ugustus) 1-2-3-#.
63. Strong, Edward K(ellogg), Jr. 0-0-3.
64. Terman, Lewis M(adison) 0-0-3.
65. Thorndike, Edward L(ee) 1-2-3.
66. Titchener, E(dward) Bradford 1-2-3.
67. Warren, Howard C(rosby) 1-2-3.
68. Washburn, Margaret F(loy) 1-2-3.
69. Watson, John B(roadus) x-2-3.
70. Wells, F(rederic) L(yman) 0-2-3.
71. Wheeler, W(illiam) M(orton) 1 (Zoology) -2 (Zoology) -3 (Zoology).
72. Whipple, Guy Montrose, x-2-3.
73. Wissler, Clark x (Anthropology) -2 (Anthropology) -3 (Anthropology).
74. Witmer, Lightner 1-2-3.
75. Woodworth, R(obert) S(essions) 1 (Physiology) -2 (Physiology) -3 (Psychology).
76. Woolley, Mrs. Paul G. (Helen Bradford Thompson) x-x-3.
77. Yerkes, Robert M(earns) x-2-3.

SUMMARY

The personnel experiment with records of American Psychologists has proved remarkably successful from the standpoints of making studies of the present status of the science of psychology and of readily furnishing useful information in accordance with the policy of the Research Information Service. The practicability of the mechanism having been definitely established and its serviceability having been demonstrated to a limited clientele, it remains to extend the Service to those who are in a position to utilize the opportunity.

By means of the Findex punch-card system, the records of American psychologists have been arranged to provide available information about different subjects of psychological inquiry. From the indexed personnel records it has been a simple task to determine the frequency of professional activity or interest in a given specialty. Similarly it has required only a few minutes' work to obtain evidence of the changes in activity at different periods. For various purposes the Findex method has proved both versatile and expeditious.

This method of exhibiting the trends of psychology contrasts in certain respects with efforts to measure degree of specialization and activity through classification of the literature for certain periods or through surveys of departmental and laboratory organization in academic institutions. An analysis of the activities of the individuals behind any movement probably offers the most promising method of evaluating its course and status. By this method preponderance of interest at present in practical phases of psychology is readily demonstrable. Change from speculative to experimental interest also is revealed. Demonstration of the frequency with which interests in psychology are combined with related fields of inquiry is another illustration of the use of the Findex file.

The use of terminology to designate specialties, even by the membership of the American Psychological Association, is in many instances quite inconsistent. There is indication of confusion in the designation of subdivisions of the field of psychology which have been roughly classified as branches, methods, and points of view. Terminology representing each of these classifications appears in the Year Book of the American Psychological Association as "subjects of research." As a result of this lack of precision in reporting subjects of research, the scatter of activities represented by American psychologists evidently appears greater than it should.

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CERTAIN PROBLEMS IN ACOUSTICS

Compiled by

The National Research Council Committee on Acoustics

CONTENTS

PREFACE

The following report, prepared by the Committee on Acoustics of the National Research Council is a consideration of thirteen selected subjects in the field of acoustics. Each of its thirteen sections has been prepared by a sub-committee appointed to consider that particular topic with care, and to present an appropriate report thereon. In assigning these topics, the guiding motive of the Committee has been a desire for the best results, rather than for a careful and correct

*This committee of the Division of Physical Sciences of the National Research Council consists of the following members: G. W. Stewart, Professor of Physics, State University of Iowa, Chairman; A. L. Foley, Professor of Physics, Indiana University; L. V. King, Professor of Physics, McGill University; D. C. Miller, Professor of Physics, Case School of Applied Science; P. E. Sabine, Riverbank Laboratories, Geneva, Illinois; F. R. Watson, Professor of Experimental Physics, University of Illinois; A. G. Webster, Professor of Physics, Clark University.

classification; hence little or no attempt has been made to avoid possible overlapping of content in the various sections. Nor is the order of presentation of either the topics, or the problems which appear under them, significant, inasmuch as the subjects have been chosen arbitrarily for convenience, and are merely those that, up to the present time, have been considered in detail by the Committee. In a few instances, instead of presenting specific problems under the particular topic, the status of the whole subject has been sketched, and the general nature of its chief problems indicated. At the opening of each section are given the names of those members chiefly responsible for the ensuing discussion.

I. AUDITION

SABINE, STEWART

Much of the work heretofore done upon the physical characteristics of the ear has been done under circumstances in which the physical aspects of the problem have not been sufficiently considered. Thus experiments have been conducted under conditions varying so widely that results are scarcely comparable. While many questions under the general subject of audition call for further study, those of more immediate interest may be grouped under two heads:

I. Characteristics of the ear in absolute dynamical units. II. Physical aspects of sound localization.

Under I, the following facts are needed:

(1) The value in absolute units of the minimum intensity necessary for audibility over the range of frequencies used in music and speech.

(2) The relative intensities for equal loudness sensation of tones of different frequencies, comparison being made at different intensities. Results obtained heretofore by different observers using different methods are not in agreement.

(3) The minimum intensity change necessary for a perceptible change in loudness as a function of both frequency and loudness. An approximately logarithmic relation has been found to exist between intensity and loudness, for a single frequency, but the work calls for verification and extension.

(4) The perception of a chosen sound in the presence of other sounds.

Under II, the general purpose of further study is to establish the *physical* factors entering into sound localization and, if possible, to determine their relative importance. In the first four problems proposed, sound from a single source or from two similar sources is presented to the ears in such a way that there can be introduced either known ratios of intensities at the two ears or known differences of phase, or both:

(5) Extension of the study of the binaural logarithmic intensity law.
If a pure tone from a single source is led separately to the two ears a phantom source results whose position varies with the ratio of the intensities for zero phase difference. The angular displacement in a horizontal plane from the median line is proportional to the logarithm of the ratio of the intensities, the proportionality factor for a given frequency varying with different observers and increasing with increasing frequency. For a large proportion of observers there is a lapse of this intensity effect in certain regions of frequency. The extension proposed refers specifically to the following:
(a) The entire range of frequencies, accurate tests having been made only at 256 and 512 d. v.
(b) More particularly the frequency region less than 256 d. v. and the borders of the regions at frequencies above 800 d. v. where lapses are found.

(6) Extension of the study of the binaural phase effect.
When the phase difference of the sound arriving separately at the two ears from a source of pure tone is varied, intensities being equal, the horizontal angular displacement of the phantom source from the median plane is proportional to this phase difference at the two ears. The upper limit of the effect has been established. The extension proposed refers specifically to the following:
(a) Study of this phase effect for less than 100 d. v. This frequency region has not been investigated.
(b) Verification of present evidence which shows that the constant in the linear relation does not change noticeably with individuals.

(7) Study of the combined effect in locating sound sources with simultaneous variations of intensity ratios and phase differences.

(8) Discovery of physical factors involved in binaural localization other than those described in the foregoing.

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II. ACOUSTICS IN NAVIGATION

KING

The application of acoustics to navigation is an important aspect of acoustical engineering and evidently involves contributions to the science itself. The development of both aerial and submarine generators and receptors is of obvious importance to navigation in times of peace. That the subject has received an increased impetus from the problems involved in war is true, but the reason is to be found not only in the unusual importance of the subject at such a time but also in the conjunction of availability of funds for research in this field and the lively immediate interest of scientific men. Published accounts of the application of acoustics to navigation during the world war will be found in the bibliography at the close of this section. That there is a serious need of cooperation between the government and scientific men, and between governments themselves in the establishment of extensive and costly researches is apparent. Without cooperation on a large scale, the prohibitive cost will make progress slow.

The discussion in this section will be limited specifically to fog alarm apparatus, but with an added general statement as to future problems in acoustical engineering. Certain submarine problems will be offered in the next section, the consideration of the remainder of the field being postponed.

I. FOG ALARMS

It has long been known that steam and compressed-air sirens employed as fog-alarms on land stations and on ships are extremely inefficient when the ratio of power input to acoustic energy emitted during a blast is considered. In the case of steam whistles and sirens on ships, the inefficiency is further aggravated by the large waste of live steam involved. It is only very recently, however, that the acoustic behavior of modern fog-alarm apparatus has been studied in detail, and methods devised for actually measuring the output of sound as well as its distribution of intensity at distances of several miles.

1. *Acoustic efficiency.*—Measurements of acoustic efficiency of a diaphone[2] were first made in 1913[1] and found to be 8 per cent at normal operating pressure. In 1917[1] the observations on an improved design showed a decrease in efficiency from 24 to 8 per cent, with decreasing pressure, the limits of pressure being 29 and 6 pounds per square inch.

The amount of power which it is possible to deliver to the atmosphere is limited, for the dissipation of energy in a wave of finite amplitude is an important factor. Little is known experimentally in this important field of research. Measurements to which reference has been made indicate large energy losses within 2,000 feet of the diaphone.

2. *Quality of the sound.*—The 1917[1] tests on the diaphone, made with the cooperation of Professor Dayton C. Miller, showed that the purity of the tone increases with distances. While the wind is importantly responsible for great variations in the intensity of sound, the eddy structure[3] of the atmosphere is responsible for a high degree of attenuation of a nature that affects quality. (See Section IV of this report.) From the harmonic analysis of phonodeik records it is now possible to obtain accurate data as to the relative proportions of energy contained in the master tone and in the overtones. As the master tone alone survives to an appreciable extent at distances greater than two miles, it is obvious that the designer of fog-signal apparatus should arrange to concentrate the greatest possible amount of energy into the fundamental.

II. General Statement as to Future Problems in Acoustical Engineering in Navigation

The researches just described may be considered to have established the possibility of employing methods of physical measurement in that branch of acoustical engineering. The Webster phonometer has been shown to be well adapted to the selective measurement of the master tone and to acoustical surveys by means of which the effect of atmospheric conditions on the propagation of sound may be graphically recorded. For the study of the quality of sound emitted, the phonodeik devised by Professor Miller has been proved capable of use under open-air conditions. For the measurement of acoustic output and efficiency, the thermal method devised by King seems to be satisfactory in practice. From a thermal test combined with an analysis of a phonodeik record, complete information as to the performance of a siren or a diaphone may now be obtained. For instance, it is possible to state in horse-power or watts the total acoustic output of a siren, as well as to compute the relative proportions of power contained in the master tone and in the overtones. That this may be done as a shop test is of considerable importance, since a designer will now be able to predetermine the behavior of fog-alarm apparatus without depending on tests carried out at great cost at a sea station. It is to be hoped that, with such methods of testing available, the development and improvement of fog-alarm apparatus will be much more rapid than in the past.

Considerable progress in fog-alarm work has recently been made by the Submarine Signal Company in the installation at the Fire Island Light near New York Harbor of synchronous signaling by radio-telegraph and by submarine sound. A series of dots are automatically sent out through the aether and through the water, the intervals between them corresponding to the time taken for sound to travel, say, 1,000 feet. The ship operator has his telephone connected to the aerial and to a hydrophone. If he counts the number of dots heard from his wireless apparatus before the first of those traveling by water reaches his hydrophone, he evidently obtains at once the distance from the source, reckoning 1,000 feet per dot. If his ship is also equipped with a wireless direction finder, he knows his exact location and can thus steer a safe course in fog.

It is evident from the results just described that improvement in the design of aerial and submarine sound generators offers a large field for scientific effort. Systematic research should be carried out with a view to making fog-alarm equipment (aerial, submarine and wireless) as efficient, as reliable, and as cheap as possible. With our present knowledge of acoustical and electrical engineering, it should be possible to develop a standard type of installation for use both on shore and on ships.

Research towards this objective would require a special experimental station conveniently situated by the sea, and in addition a sea-going ship specially fitted for research in *marine physics*. The initial cost, maintenance and routine management of such an enterprise can be undertaken only by a Government Department. In directing researches, the cooperation of physicists making this field their specialty should be secured. The type of equipment finally developed and tested should form the basis of specifications for the construction of shore and ship installations. In the course of time it may be expected that such installations will attain to a more or less standardized pattern of highest possible efficiency, just as various units of electrical machinery have now become standardized throughout the entire world.

Considering the large expenditure upon aids to navigation on the part of the various maritime countries, it is evident that a concentration of scientific research on these problems with adequate facilities for experimental work at sea would, in a few decades, more than repay the expenditure incurred, through reduction of the yearly toll in lives and property resulting from *preventable* accidents at sea.

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*Supplied by Dr. H. C. Hayes, of the U. S. N. Engineering Experiment Station, Annapolis, Md.

III. PROPAGATION OF SOUND IN LIQUIDS AND SOLIDS

WEBSTER, KING

The most important problem under the above title relates to the generation, transmission and reception of sound waves in water, in view of applications to submarine signalling. The subject has received a great deal of attention during the war, and much valuable information has been obtained by the research organizations connected with the navies of the various maritime powers. Except in a few cases, however, the results of these researches have not been made public.[1]

The only type of sound generator for submarine work generally available is the Fessenden Oscillator.[2] The essential part of this apparatus, which weighs about 1300 pounds, consists of an extremely ingenious *linear induction motor* invented by Professor Fessenden. By its use an alternating current of 500 cycles is induced in a *vibrator* in the form of a copper cylinder which, reacting with an intense magnetic field, transmits a powerful *vibro-motive force* to the center of a heavy steel diaphragm one inch thick and about 30 inches in diameter.

I. PROBLEMS CONNECTED WITH THE DESIGN OF SUBMARINE SOUND GENERATORS

In order to design an electromagnetic sound generator to give a maximum efficiency at resonance for a minimum weight, it is necessary to know the acoustic load on the diaphragm and the size of the diaphragm which will give maximum radiation at resonance frequency. Research in this direction is best carried out by estimating the acoustic load from theory and making this the basis of design. From a determination of the electrical constants of the completed apparatus and a knowledge of the internal losses, it is possible to estimate the acoustic output of the diaphragm when radiating in water. It is important that such tests be carried out in deep water, tank experiments leading to erroneous results owing to internal reflections.

By a combination of theory with the result of electrical measurements, it is possible to arrive ultimately at a reliable knowledge of the acoustic characteristics of sound-generating diaphragms. As far as the writer is aware, there is no published information available in this important field of research.

The acoustic radiation field in the neighborhood of a powerful sound generator calls for investigation with a view to comparison with theory. An important practical problem is to determine the maximum power which it is possible to radiate across a square centimeter under given conditions of pressure. As is well known, the water breaks down, or *cavitation* sets in over the diaphragm when the amplitude exceeds a certain value, depending on the depth of immersion. In the radiation field close to the diaphragm there is a tendency for small bubbles of dissolved air to form near solid objects and around suspended dust particles, owing to the fact that the rarefaction in the sound waves may exceed the static pressure.

II. Problems Connected With the Design of Submarine Sound Receivers.

Research on hydrophones for picking up underwater sounds received a great deal of attention during the war in connection with anti-submarine problems. In view of their importance in ordinary navigation it is to be hoped that investigations will be continued in this important field.

The following items of research still require to be carried out:

(1) A thorough experimental investigation of the characteristics of circular diaphragms vibrating in water with the object of determining (a) the acoustic mass, (b) the damping due to acoustic radiation, and (c) the form assumed by the vibrating diaphragm. The theory of a circular diaphragm vibrating in water has recently been published by Professor Lamb,[4] but experimental confirmation is still lacking.

(2) The study of diaphragms of different materials with a view to determining the damping due to elastic hysteresis.

(3) Determination of the most sensitive forms of microphone contacts for the reception of underwater waves, involving a minimum dissipation of energy.

(4) Determination of the most suitable and compact form of electron-tube amplifiers to be used for long-distance reception.

(5) The improvement of telephone receivers used in reception. The best types are known to have very low efficiencies even at resonance.

(6) The substitution of visual (or graphic) reception for telephonic reception, and the calibration of such receivers as secondary standards of sound measurement.

(7) The elimination of extraneous noises in reception from a moving ship.

(8) The determination of the maximum distance at which submarine signals of constant pitch can be determined, making use of a tuned receiver of maximum sensitivity with highest possible amplification.

III. Transmission of Sound in Solids

An important practical problem under this head is the study of the transmission of sound through partitions employed in building construction. This subject is now receiving attention at the Riverbank Laboratories, Geneva, Illinois, under the direction of Dr. Sabine.

The transmission of sound through earth and rock received considerable attention during the war in connection with military problems. With the exception of a very incomplete and fragmentary paper by the writer and Dr. A. N. Shaw,[5] reviewing published work up to the year 1915, and describing a few new experiments, there is little information available.

During the war extremely sensitive "geophones" were devised for detecting mining operations in trench warfare, but no reliable information on these instruments seems to have been published.

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IV. PROPAGATION IN THE ATMOSPHERE

Foley, Webster

Theoretical investigations of the propagation of an acoustic wave in the atmosphere (Milne, Phil. Mag., p. 96, July, 1921, and others therein mentioned) assume that the atmosphere is a stratified medium, the layers differing in velocity of propagation of sound and in wind. These investigations give the corrections to be applied to the bearing and elevation of sound located acoustically, and discuss the range of audibility as limited by total reflection. The formulæ obtained should be checked by trial to note the practical limitations of their applicability. But the serious influence of atmospheric conditions, presumably a variation from planar stratification, upon the range of audibility (Stewart, Phys. Rev., xiv, p. 376, 1919, and Taylor, Phil. Trans., vol. 215A, pp. 1–26) and the variation of this range with frequency demands an approach by means of an experimental study. The fog-signalling engineers are directly interested in such a study and there is in atmospheric propagation a field for important investigation, though requiring an elaborate equipment which can best be supplied through cooperation of the government with a scientific group.

V. REFLECTION, ABSORPTION AND TRANSMISSION AT THE SURFACE OF AND WITHIN CERTAIN MATERIALS

SABINE, WATSON

Sound waves in air impinging upon another medium are reflected, absorbed, and transmitted in varying amounts, depending upon the physical characteristics of the material of the medium and its structure. If channels or pores exist, there is dissipation because of viscosity and heat conduction. There are also inelastic flexural yielding of the reflecting surface, inelastic compression, and inelastic flexure of the minute structural parts of the material. It is thus not easy to interpret experimental results on reflection, absorption and transmission, and theoretical considerations have peculiar difficulties. The accompanying statement of specific problems in the first two instances presents aspects chiefly of scientific interest. The third states the most desirable next development in the applied field of architectural acoustics.

(1) A theoretical solution of the problem of the reflection of a compression wave from a non-elastically yielding surface should be developed to account for the shape of the absorption curves already obtained, and to predict absorption coefficients from mechanical properties.

(2) A study should be made of the reflection, absorption and transmission of sound by various materials under ideal conditions in which the effects of the different factors may be separated and their magnitudes determined separately. This study should include a consideration of the physical properties of density, stiffness, damping and porosity with a view to the development of a suitable mechanical theory.

(3) The study of the transmission of sound through actual building constructions, the tests being carried out under conditions approximating those found in standard practice. In these tests, distinction should be made between the case in which the sound passes from the air through the partition, and that in which the source of sound is in intimate contact with the structure. A study of the various sound-deadening materials and the conditions for their most effective use should be included.

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VI. THE MEASUREMENT OF SOUND INTENSITY IN ABSOLUTE UNITS

SABINE, WEBSTER

The complete theoretical and experimental solution of this problem calls for an extended investigation. It involves the production of a constant and reproducible source of sound, of determinable acoustic output, a measuring instrument whose readings may be reduced to absolute dynamical units, and means for varying sound intensity in a known way. Up to the present, these three aspects of the question have never been thoroughly investigated in a single experimental research. A brief summary of the various modes of attack heretofore used on these different parts of the general problem follows:

I. METHODS OF MEASURING ACOUSTICAL OUTPUT

(1) The density variation in a vibrating air column has been determined by measuring the shift of the interference fringes formed by two beams of light, one of which traverses the vibrating column. The fringes were viewed stroboscopically and the sound output computed from the theory of open pipes.

(2) The pressure variation in a sounding organ pipe has been measured by means of a delicate manometer and a synchronously operating valve. Obviously the method is limited to sounds of low pitch and great intensity.

(3) The acoustical output from telephonic sources has been determined. The method calls for an analysis of the diaphragm motion as a function of the electrical input, by actual study of this motion, using optical methods under varying conditions of current amplitude and frequency.

(4) The Phone, developed by A. G. Webster, is an instrument whose output may be computed from easily made measurements. It consists of a cylindrical resonator capable of tuning over a considerable range of frequency, with a tuned diaphragm mounted in the resonator opening. The latter is supported by three wires, and moves as a whole when actuated by alternating current of the same frequency as that of the resonator and the diaphragm. The amplitude of vibration is measured microscopically.

(5) The Thermophone has been suggested and used as a standard source of sound of easily calculable output. The passage of an alternating current through a very thin strip of metal foil produces rapidly damped temperature waves in the air near the metal strip. The

expansion and contraction of the air results in the generation of a sound wave whose intensity may be computed from the electrical and thermal coefficients of the strip, its dimensions, and the value of the alternating current.

II. Methods of Absolute Measurement

Various methods have been used for the comparison of sound intensities. Some which are susceptible of giving results in absolute units are as follows:

(1) The determination of the static pressure due to a train of waves gives immediately a value of the energy density of the train. The method has been employed in the analogous case of light. Such a method would be applicable only to sounds of very high pitch and short wave length.

(2) Wien and others have employed a form of vibration manometer. The corrugated diaphragm of an aneroid barometer is made part of the inner wall of a resonator tuned to the frequency of the sound to be measured. The motion of the diaphragm under the varying pressure of the sound was measured by the width of a band of light reflected from a small mirror, one point of which was in contact with the diaphragm and which rotated with the motion of the latter.

(3) The Condenser Transmitter has been proposed as a means of measuring in absolute dynamical units the pressure variations of sound. It is essentially an electrostatic telephone transmitter, the diaphragm constituting one plate of a variable air condenser. The fixed plate is very close to the diaphragm, the air film intervening being so small as to be an important factor in the diaphragm damping. The motion of the diaphragm varies the capacity of the condenser and produces a minute alternating current in the circuit. The latter is amplified by means of vacuum tubes to give measurable currents with a vacuum thermocouple and microammeter. The instrument must be calibrated in absolute units. The thermophone has been used for this purpose. Both instruments are in a closed space, the pressure changes being computed from the alternating currents in the thermophone that produce the sound.

(4) The Rayleigh Disc and Resonator has been used successfully for the determination of relative values of sound intensities. If the modulus of torsion of the fiber supporting the disc be known, the absolute value of the alternating air stream velocity may be determined from the deflection produced by the sound. This gives the intensity at the disc and within the resonator. For the determination of the absolute value of the sound as unaffected by the presence of the resonator,

correction must be applied for the effect of the suspended disc upon the motion of the air within the resonator, and for the ratio of the intensities inside and outside such a resonator.

(5) The Phonometer is in all respects similar in construction to the Phone already described. The diaphragm is tuned to unison with the sound to be measured, by varying the tension in the three supporting wires. Instead of using the microscope for measuring the diaphragm motion, however, a small concave mirror is caused to rotate by this motion. The width of the band of light caused by this motion measures the diaphragm motion. Theoretical treatment shows that the pressure amplitude of the air motion is proportional to the amplitude of the diphragm. The proportionality factor involves the mechanical properties of the diaphragm, its dimensions and the dimensions of the resonator, the frequency of the sound and constants of the medium. The construction of the instrument is such that all the necessary data for the reduction of the instrumental readings to absolute units of pressure on the diaphragm are determinable.

(6) If a very fine wire, heated to a dull red incandescence, be exposed to sound waves, there will result a periodic change of resistance. The wire may thus act as a receiver and the measurement of the alternating currents produced in the secondary of a transformer may be used as a basis for the comparison of sound intensities. The method has not been employed for absolute measurements, but it possesses possibilities of being adapted to this purpose.

III. Means of Varying Sound Intensities in Known Ratios

In order to check the readings of any instrument intended for absolute measurements, it is desirable to know the output of the source of sound and to be able to express the intensity at any point of the sound field in terms of the output and the distance from the source to the measuring instrument. This involves a knowledge of the distribution of sound intensity in the field. The ideal field for this experimental verification of the intensity as given by various instruments would be that of a spherical wave propagated in a homogeneous atmosphere. In general, conditions out of doors do not approximate this ideal, because of the disturbing influence of the ground and the acoustical inhomogeneity of the atmosphere. Experiments in closed rooms must take account of the marked variation in intensity due to interference in the room. In case the experiment is conducted within doors this aspect of the problem must be carefully considered. In a room of simple geometrical form, it is possible to compute the intensity at any point from a knowledge of the output and the position of the source. The theoretical solution of this part of the problem should be checked by a careful acoustical survey of the room with an instrument capable of giving relative intensities.

The foregoing serves to indicate the present status of the general problem and some of the means at hand for its solution. From the very nature of the case it is scarcely possible to point out specific problems under the general head. The need is for a single carefully organized research in which results with different instruments and by different methods are secured under conditions so nearly identical as to make these results comparable. The problem is peculiarly fundamental for real progress in experimental acoustics.

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VII. DETECTION AND MEASUREMENT OF SOUND

MILLER, WEBSTER

Probably the instruments available to the physicists for the detection and measurement of sound are less satisfactory than those for any other field of research. A wide variety of methods have been proposed and more or less developed, such as,—vibrating membranes and diaphragms, applied as in the phonautograph (phonograph), the phonometer, and the phonodeik; vibrating diaphragms in connection with electrical effects as in the carbon transmitter, the condenser transmitter, the thermophone; the Rayleigh disk; the hot-wire anemometer; sensitive flames; and the direct detection of waves in a medium by the "refractometric" method.

For the measurement of the energy of sound the methods are quite limited in number and in applicability. These are: the Rayleigh disk, the phonometer, the sound-pressure balance, and indirect methods by the use of electrical apparatus.

For the making of records of sound waves for quantitative investigation, various methods of detecting sound may be combined with some form of recording apparatus, usually photographic in nature. Among those available are: various miscrophones and transmitters in connection with the string-galvanometer or oscillograph, the "barreter," the phonodeik, and the refractometric method.

Under this section the needs involving research are many and urgent. In general these needs are:

(1) Sensitive instruments for detecting and measuring sounds, the instruments being free from all resonance effects which distort the record, and being capable of responding to the entire range of frequencies.

Specifically, some of these needs are:

(2) A direct-reading instrument capable of determining the absolute (and relative) intensity of a sound, equivalent to the photometer in optics.

(3) An instrument for recording sound waves in air, as in the phonodeik, but free from the disturbing effects of the horn and diaphragm.

(4) A method for directly photographing the sound waves in air, as in the refractometric method, but directly applicable to common sounds as to amplitude and frequency.

(5) Reference standards of sound intensity for various frequencies, and of various multiples of the unit.

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See also the lists of references on the *Measurement of Sound Intensity*, Section VII; on *Sound Generators*, Section VIII, and on *Photography of Waves*, Section XI, of this report.

VIII. THE EFFICIENCY OF SOUND GENERATORS

WEBSTER, KING

Only comparatively recently has it been found possible to estimate the *acoustic efficiencies* of sound generators. In the following sections are briefly enumerated various methods by which the acoustic efficiencies of sound generators have actually been determined.

I. COMPRESSED AIR SOUND GENERATORS

When compressed air generates sound waves in any manner, mechanical work is done, represented by the energy propagated away as sound. As a result there is a drop in temperature as the air issues from the high to the low (atmospheric) pressure. The temperatures are supposed to be measured in regions of stream-line flow free from eddy-motion. It may be shown[1] that the acoustic output is given by the formula

$$\dot{w} = JC_v\dot{M}\,(T_1 - T) \qquad (1)$$

where J is the mechanical equivalent of heat, C_v, the specific heat of air at constant volume, $\dot{M}$ the rate of mass-flow of air, T_1 the absolute temperature of the air on the high-pressure side, and T the absolute temperature on the low-pressure side.

In an ideal sound generator consuming air at the same rate and working on an adiabatic cycle, the acoustic output is

$$\dot{W} = JC_v\dot{M}T_1\left[1 - (p_0/p_1)^{\frac{\gamma-1}{\gamma}}\right] \qquad (2)$$

where p_0 is the atmospheric pressure, p_1 the operating pressure, and $\gamma=1.41$, the ratio of specific heats for air.

From (1) and (2) is obtained for the acoustic efficiency the following simple formula,

$$\eta = \frac{\dot{w}}{\dot{W}} = \frac{T_1 - T}{\left[1 - (p_0/p_1)^{\frac{\gamma-1}{\gamma}}\right]T_1} \qquad (3)$$

in which all the quantities which enter are easily measurable. In high-powered acoustic apparatus, the temperature difference $(T^1 - T)$ may amount to several degrees C., and is easily measured by means of a portable resistance thermometer or thermocouple. The application of this method to the study of wind instruments should be made the subject of further investigation.

It may be noted that the thermal method just described is independent of the frequency, and is therefore suited to a determination of the total acoustic output of a sound generator, which determination includes the energy contained in all component tones. The relative amounts of energy contained in the overtones may be determined by photographing the sound-wave by means of Professor Miller's *"phonodeik"* [2] and analysing the corrected wave-form.

II. Electrically Operated Diaphragm Generators

The diaphragm is the essential feature of a number of sound-reproducing instruments (including phonographs and telephone apparatus), while electrically operated, heavy steel diaphragms are employed to generate sound in such a relatively incompressible medium as water.

It may be shown from theory that the rate at which radiation is emitted from a single diaphragm of dimensions small compared to the wave-length, vibrating in an infinitely extended medium of density ρ is [3,4]

$$\bar{W} = 2\rho\pi^3 f^4 V^2/c \qquad (4)$$

where

$$V = \int v\, dS$$

is the vibration-amplitude v integrated over the area of the diaphragm, and f is the frequency. The velocity of sound in the medium is denoted by c.

For diaphragm instruments vibrating in air, V is determined by integrating the experimentally determined distribution of amplitude over the area of the diaphragm. A method of doing this has been worked out by Kennelly [5] for the diaphragms of ordinary bi-pole telephone receivers vibrating over a large range of frequencies. King [6] has applied formula (4) to these observations and has obtained for the efficiencies, values ranging from six parts in a million in the neighbourhood of 400 cycles to four parts in a thousand at a resonance frequency near 1000 cycles. It is evident that the improvement of telephone receivers designed to give higher efficiencies constitutes an important field of research.

Equation (4) enables us to construct sound generators of known acoustic output. By using diaphragms tunable by air pressure,[7] an interesting field of research lies in the design of self-operating standard sound generators of variable pitch.

The measurement of the distribution of amplitude of a diaphragm vibrating under water is a matter of some difficulty. In such cases the best procedure seems to be to work out the theory of the electromagnetic mechanism operating the diaphragm and to determine experimentally the internal losses so that the acoustic output may be obtained from the measured electrical input. Little actual research along these lines has been carried out.

III. Other Methods of Measuring Efficiencies

Other methods of measuring the acoustic output of sound generators are described in Section XI of this report, dealing with the measurement of sound intensity.

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IX. SOUNDS OF MUSICAL INSTRUMENTS AND SPEECH

MILLER

The physical nature of the simple vowel sounds has been studied by many investigators; however, owing to differences of pronunciation and of method these results are difficult of comparison. A very little quantitative work has been done upon the tone qualities of the organ-pipe, the flute, and the violin. But so varied and important are the sounds of music and of speech that the little already accomplished is insignificant, and the entire field may be considered as open for study. The comparatively recent development of instruments and methods for the analytical study of sound waves makes it possible to undertake such work in many lines. The development of new apparatus as outlined in the report of Committee X will greatly facilitate the work mentioned in this present section.

The subjects mentioned below are suggested as suitable for immediate investigation. It may be noted that these problems are intimately related to those on audition, architectural acoustics, experimental psychology, and phonetics; by including these aspects of the subject a very much longer list of problems could be made.

(1) The complete analysis of the characteristic sounds of all kinds of musical instruments and of their combinations as used in music.

(2) The study of the vibratory motions of the ultimate sources of sound waves.

(3) The study of the effects of sound-bounds, resonators, and all other modifying influences upon the sounds as emitted by actual musical instruments and other sources of sound waves.

(4) A further study of the vowels under standardized conditions, and an investigation of the nature of "mixed" vowels and of vowels in various languages nearly but not quite alike.

(5) A determination of the exact nature and effect of the consonants, about which very little is now known.

(6) The influence of various characteristic and attendant noises upon the perception and interpretation of sounds of speech and of music. In other words: To what extent is the identification of the source of sound dependent upon characteristic noises rather than upon simple tone quality?

(7) An investigation of the nature and effect of the flow of sound in words, of rhythm, accent and poetry.

(8) A determination of the differences in the nature of vocal sounds as enunciated in speech and in song.

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X. ANALYSIS AND SYNTHESIS

MILLER

The direct analysis of a sound has been accomplished only in a very incomplete way. Helmholtz devised the resonator to assist the ear in detecting any predicted tone, one at a time. Koenig arranged a set of resonators each with a manometric-flame indicator, for studying eight or ten components. The Rayleigh disk made the method much more sensitive and more quantitative, and recent investigators have used a series of Rayleigh disks simultaneously. The direct synthesis of sound was carried out by Helmholtz with tuning forks and adjustable resonators, by Koenig with the multiple wave-siren, and by later experimenters by means of electrical generators and the telephone. These methods are of limited application.

The analysis and synthesis of sound generally means the analysis and synthesis of the records of the wave-form of the sound. The available methods for the analysis of such records are restricted to periodic curves, that is, to curves to which Fourier's Theorem is applicable. The mechanical methods for the analysis and synthesis of periodic curves are simple and direct in operation, and are so efficient and precise that little more is desired.

The specific needs are:

(1) A method for the direct analysis of sound, similar to spectrum analysis in optics.

(2) A method for determining the components of non-periodic curves, that is, of curves whose components are incommeasurable.

(3) An apparatus for the direct synthesis of simple tones of any given frequency and intensity.

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XI. PHOTOGRAPHY OF SOUND WAVES

FOLEY

There are three general reasons why further work on sound-wave photography is desirable. One is that the usual theory of acoustics is based upon the assumption of infinitely small displacements, a condition frequently not even approximated. Sound-wave photography is most successful with waves of relatively large displacements for which a complete mathematical theory has not been developed. Further, considerable theory already developed remains to be verified experimentally, and experimental results must be obtained for cases for which mathematical solutions have not been found.

A second reason for further work on sound wave photography is that such photographs may be of considerable use to teachers and students of wave motion. Several interesting cases of sound pulse reflection, refraction, and diffraction remain to be photographed. The work already done with sound pulses should be repeated with wave trains.

A third reason is that the methods used in photographing sound waves promise to give us insight into the mechanics of sound production by electric sparks, explosions, etc., and possibly increased efficiency of sound generators and receivers. In general it would appear that the methods might be advantageously used in the solution of many problems having to do with fluid motions and disturbances but which are outside the field of acoustics.

Some of the more specific results to be sought for are stated in the following list of problems:

(1) Using a train of waves of determinable intensity and frequency (necessarily high if the photographic method is used) the velocity of the individual waves should be measured and related to distance, density variation and change in regime.

(2) The velocity of sound as determined by means of stationary waves in tubes and for progressive plane waves of the same intensity and frequency should be measured and compared.

(3) For the purpose of improving the design of mouth-pieces, trumpets, horns, and other sound-amplifying devices, an extended study should be made of the form and intensity of sound waves passing through them. The study should include the determination of nodal points, end correction, etc., of pipes, horns, and resonators of various forms.

(4) A study should be made of the actual form of a spark wave *at the spark* and how it is produced. This could be extended to include sparks between electrodes of different shapes, sizes and materials and in gases of different densities.

(5) Using the methods of sound wave photography, an extended study should be made of explosions. This should include a study of

the mechanics of the explosion, the production, forms and velocities of explosion waves and of the resulting sound waves, the effects of pressure and temperature variations, etc. The sound wave photographic method, when applied to the study of explosions, yields a series of pictures showing something of the various stages in the life history of the explosion. Photographic methods used previously yield merely a composite of the luminous phenomena produced by the explosion. It would seem that the sound wave photographic method opens a wide field to those interested in explosion engines, ordnance, etc.

(6) The method used in photographing spark waves may be used to advantage in the study of fluid motions and disturbances of many kinds, as for instance, the stream lines about bodies moving in fluids and fluids flowing around obstacles, gases streaming through orifices as in carburetors, exhaust valves, etc.

(7) Psychologists and medical men have requested a photographic study of the action of the external ear and of the ear drums of mammals and birds to aid them in their study of the mechanics of hearing.

(8) There have been requests from scientists engaged in the study of silent electrical discharges, leakages, coronas, etc., for a photographic study of these phenomena under a variety of conditions.

(9) Sound wave photography is being used with considerable promise in the study of architectural acoustics.

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XII. SOUND WAVES OF FINITE AMPLITUDE

WEBSTER, FOLEY, STEWART

The usual differential equation for the propagation of sound in air, according to which a plane wave is propagated unchanged in form, is secured by neglecting certain terms, this procedure being based upon the assumption that the amplitude is very small in comparison with the wave length of the sound. In cases where the assumption is not justified the amplitude must be considered as finite. Much theoretical work has been done in this field of "finite amplitude" but inherent difficulties have limited the results as indicated in the first problem given below. The present status of the subject indicates the need of experimental work, having as its purpose not verification of theory but rather the addition of facts which theory should attempt to make clear. These experiments are indicated in the other problems here cited.

Specific problems are:

(1) The solution for spherical and cylindrical waves of finite amplitude should be obtained. The solution for plane waves has already been found, but such waves are difficult to obtain experimentally. Spherical and cylindrical waves can be approximated much more closely and would give a better opportunity for comparison of theory and experiment.

(2) Measurements of the pressure and density variations in sound waves from guns and from electric sparks and simultaneous measurements of the velocities of these waves should be made. It would appear that experiment in this field should precede additional theory with the object of supplying the facts upon which a theory can be constructed.

(3) Similar facts should be obtained in regard to waves of permanent regime such as the nose wave of the bullet. The purpose stated for the problem just preceding exists here also.

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XIII. CONICAL HORNS.

WEBSTER, STEWART

Theories of the action of conical horns are unsatisfactory for the reason that they assume spherical waves in horns of "finite" angular aperture, or plane waves in horns of very small angular aperture. Thus no account of natural vibrations, other than radial, are considered. An extension of the theory is therefore important. Recent experimental results suggest the advisability of the extension of our knowledge of the action of conical horns. The following are specific problems:

(1) The theory of conical horns used as receivers, taking into consideration cross-sections whose diameters are comparable to wave lengths should be obtained. This extension is needed to ascertain, if possible, the cause of the existence of the optimum angle of a horn, judged by the amplification produced, and its variations with frequency and order of overtone.

(2) The amplifications produced by conical horns used as receivers, for a wide range of frequencies and wave lengths should be measured. Present theory is not sufficiently exact to enable one to make quantitative predictions as to the amplification at the apex of the cone and published data are too meager.

(3) The corrections experimentally found necessary to compute natural frequencies from a knowledge of the dimensions of conical horns should be ascertained. The chief interest of such facts is in connection with the theory, though there is a practical value of the data as well.

(4) The modification in the "end correction" when cones of differing angles are connected in series should be ascertained. These results are also of a practical and theoretical interest.

(5) The position of the nodal surface on the interior of a receiving conical horn when the frustum is gradually extended to the apex and the resonance frequency is used should be found. The object of the data is chiefly utility in exposition, since a clear statement of the action of a conical horn is not published even in approximate terms.

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ELECTRODYNAMICS OF MOVING MEDIA

Report of the National Research Council Committee on Electrodynamics of Moving Media

By

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PART I

THE FUNDAMENTALS OF ELECTRODYNAMICS

BY W. F. G. SWANN

INTRODUCTION

The present report purposes to make a critical survey of some of the more familiar aspects of the subject with a view of inquiring as to what parts may be considered as relying upon experiment, or upon deduction from fundamental principles, and what parts are mere definition or scaffolding, convenient for the correlation of the experimental laws but in no sense calling for experimental proof themselves. Any individual experiment usually covers a very limited region of the whole field of investigation of which it forms a part. It is a representative of a much larger class, from which it is chosen on the grounds of practicability. In citing an experimental foundation in any particular case we shall not therefore confine ourselves to individual experiments which have been performed, but shall speak rather of general classes of experiment of which any particular experiment will form a special case.

One method of procedure would be to take the complete scheme of equations as formulated by Lorentz, and discuss everything in terms of these. To do this, however, would be to clothe the subject with considerable artificiality as regards certain important fields of its application. The attitude which one adopts toward the subject, his way of thinking about it, even the method of expression of its laws is to some extent bound up with the degree of generality which he has in mind for its application. Thus, for example, there is a perfectly consistent scheme of hypotheses applicable to the case where one confines himself to unvarying currents in closed wire circuits; and while this subject may be treated as part of the general case of electronic motion, it may as far as its own requirements are concerned, be discussed in different and somewhat simpler language. The field of application of this special case is moreover so great that it constitutes as it were a little subject in itself, and it is therefore convenient to discuss what would be its laws from the standpoint of one who never had occasion to deal with more general cases. Following out this plan, we shall consider the whole subject in four stages applicable respectively to the following cases.

1. Electrostatics.
2. Steady currents in closed circuits.
3. Varying currents in closed circuits.
4. The general case as symbolized by the Lorentzian equations for free aether and their adaptation to a material medium.

Certain matters of detail will be relegated to notes in an appendix to this report.

Electrostatics

The laws of inverse squares and of conservation of charge.—The fundamental law is here, of course, the law of inverse squares; but, the meaning of this law is to some extent a function of the light in which we regard it. We may speak of it in terms of the law of force between certain material bodies, and we may in fact formulate the general law in the following way:

> Let there be given an assemblage of bodies. Then it is possible to assign to each element of volume *of the bodies* a definite number ρ such that on writing down vectors
>
> $$\rho_1\rho_2 d\tau_1 d\tau_2/4\pi r^2$$
>
> for the forces between each pair of corresponding elements, the forces so obtained when compounded will give the resultant mechanical forces and couples exerted by the bodies on each other.
>
> Moreover, if the bodies be moved to new positions, a similar result holds, and the values of $\int\rho d\tau$ for any one of the bodies in the two positions are equal. } A

Surface distributions may, of course, be included as limiting cases of volume distributions.

This is the law which may be regarded as the experimental basis of electrostatics. In this form the definition of charge density is not made until the law has been recognized, and indeed until then it has no meaning. The superposability of effects, the proportionality of mechanical force to charge, and the conservation of charge are contained in the statement of the law, once the experimental fact has been assumed.

A consideration of the matter will show that all electrostatic experiments designed to test the law of inverse squares are particular cases of a general experiment of this type. An experiment with Coulomb's torsion balance is of course of this type; and while the statement may not be obvious at first sight, a little consideration will show that even experiments having to do with the absence of field inside a hollow conductor are of this form. Thus in its most general form this experiment with the hollow conductor may be taken to show that a charged body placed inside such a shield experiences no mechanical force as a result of the presence of charges outside. Now it is a mathematical fact that whatever distribution ρ be assigned to the space outside of a surface, it is always possible to assign a function σ varying over the surface in such a way that while $\int\sigma dS=0$, the vectorial sum of

$$\iiint\frac{\rho d\tau}{r^2} \quad \text{and} \quad \iint\frac{\sigma dS}{r^2}$$

is zero for any point which is inside the surface, and whose distances from the various volume elements $d\tau$ and surface element dS are given by the corresponding values of r. It follows that if the mechanical force upon a body is given by a law of the type above specified, it is certainly possible to assign a distribution σ which when combined with the distribution ρ will result in zero mechanical force on a charged body within the surface, and in the limit on the surface. The fact that there are material surfaces (conductors), whose nature is of a kind such that the distribution σ will automatically come about is something which is outside the scope of assertion of the fundamental law. Once experiment has revealed that there *are* surfaces inside which a body will experience no force although bodies outside that surface attract and repel each other, we have realized a situation which certainly can be accounted for by a law of the type A preceding, and it is possible to prove with a fairly large degree of generality that a law of this type is the only one possible to account for the phenomenon.

In the above formulation of the law, no mention of a field is made It is, however, a mathematical convenience to *define* now a quantity E as

$$4\pi E = Grad \iiint \frac{\rho}{r} d\tau. \tag{1}$$

This definition involves nothing about the force on a unit charge, and it is automatically contained in the experimental law as above stated that the force on an element of matter charged with density ρ is $\rho E d\tau$.

It may be parenthetically remarked that the possibility of relegating E to a minor role as regards the expression of the law is not one which is confined to the statical case, but can be employed with advantage even in the most general case, and is indeed the only logical way of formulating the laws in a manner capable of experimental proof. Thus, for example, nobody can assign any meaning to the field inside a moving electron when that field is defined as the force on a unit of charge attached to a piece of matter of macroscopic size. In the light of the above statement, one might question the necessity for introducing such a quantity as E at all. It is true that E need play no fundamental role as regards the expression of the fundamental laws; but, there are a number of facts, concerned with the special properties of materials, which are of enormous importance from the practical standpoint although they are not to be regarded in the light of fundamental laws. Such a fact is the existence of a body possessing the properties of what we call a conductor. A conductor is a body in which the charge distribution must adjust itself in such a way that the quantity E as above defined will always be zero within its substance. It is thus in terms of this subsidiary quantity E that this non-fundamental but very valuable property of a conductor becomes expressed.

In spite of what has been written above concerning the advantages to be derived from relegating E to a subsidiary role in the matter of expression of general laws, one must face the fact that it is customary to speak of a quantity E which is regarded as defined in terms of the usual unit charge, a convenient silence being maintained as regards the realization of this definition in such a case as that of a point inside an electron for example. Without therefore attempting to justify this attitude, or even to remove the logical difficulty in respect of such an imperfect definition, it is of interest to consider what becomes of the law of inverse squares for a case where our starting point is the hypothesis that E is defined or is capable of definition at any point in space.

Case where electric field is the quantity fundamentally defined.—The first thing that is necessary is a definition of charge density. Charge per unit volume would only move matters a step farther back, by calling for a definition of charge. Without belaboring this point unduly we may recall that the necessity of making a definition of ρ which is applicable in all cases, including that of a moving charge, has resulted in its being defined in terms of the field as

$$\rho = Div\ E \tag{2}$$

With this definition as applied to the statical case, however, a rather remarkable result follows.

It is a result of pure mathematics, and independent of experiment, that any vector E can be expressed at a point P in the form

$$4\pi E = -Grad \int\int\int \frac{Div\ E}{r} d\tau + Curl \int\int\int \frac{Curl\ E}{r} d\tau \tag{3}$$

where the volume integrals are taken throughout all space, and r is the distance of an element of volume from the point P.

If we make the assumption that the vector E has no curl, and we may take this as our definition of what we mean by limiting ourselves to an electrostatic system, we have

$$4\pi E = -Grad \int\int\int \frac{Div\ E}{r} d\tau \tag{4}$$

If ρ is defined as $Div\ E$, this is nothing more nor less than the expression of the law of inverse squares for the entity whose density is given by ρ. If, for example, we have a point 0, and we assign a field which varies inversely as the cube of the distance from 0, it is obvious that the field in question will give rise to definite values of $Div\ E$ throughout space, and the above mathematical theorem shows that the amount of this is just such that the field may be regarded as one of the inverse square type, produced however by a distribution of charge which is not confined to the point 0.[1] It is useless to say that there is no charge save

[1] The infinite charge which will be necessitated at 0 may be avoided by specifying the inverse cube law as applying only outside of a sphere of finite radius.

at 0, for in terms of *Div E* as the definition of ρ, the very act of assigning such a field as one varying according to the inverse cube will automatically carry with it the assignment of the charge distribution necessary to represent this field as consistent with an inverse square law. These remarks are particularly pertinent to such cases as those where it is customary to speak of the forces between electrical charges in atoms as obeying laws of force other than the inverse square, even in the statical case. Such laws need be regarded as in no sense inconsistent with the classical theory, and indeed cannot be.

What then remains of the experimental proof of the law of inverse squares for the type of formulation which defines ρ as *Div E*? We must ask what it is that we wish to prove. The most reasonable answer is that we wish to prove that there are certain conditions in which the field in a given region can be expressed entirely in terms of fields due to charges outside of the region, which charges act according to the inverse square law. From the standpoint we have adopted, however, this result will follow as a consequence of pure mathematics provided that we have *Div E* zero throughout the region. The whole experimental law can thus be formulated in the statement that there are regions in which *Div E* is zero.

Our search for a region in which *Div E* is zero must of course be made in the light of a precise definition of E. As regards phenomena on a sufficiently large scale the usual definition in terms of unit charge will suffice. We now observe that if we can find a case in which E is zero throughout a region, (although not of course zero everywhere outside, for this would give a trivial result), we shall have found a region in which *Div E* is zero. Now experiment shows that there are regions, hollow conductors, in which we can have E zero while it is not zero outside; and from the standpoint which we are now considering, it would appear that all that this experiment of the hollow conductor may be taken to establish is that there are regions in which *Div E* is zero while it is not zero outside. We have apparently not made much use of the fact of the boundary of the region's being a conductor. Indeed the only essential thing is the *existence* of a region in which E and consequently *Div E* is zero. It is however possible to go on and show that if such a region exists, its boundary must be an equipotential surface, although this is not a fact of which the proof makes direct use.

Although the definition $\rho = Div\ E$ would carry with it the law of inverse squares for the field in the foregoing sense, an extra hypothesis to the effect that the mechanical force on an element of matter is proportional to the charge on it is necessary; for, the definition of E as the force on a particle of matter containing unit charge does not carry with it as a necessary consequence, the proportionality of the force on the matter and the charge when the latter is defined in terms of a density given by *Div E*. The assumption of the said proportionality would therefore be one subject to the necessity of experimental investigation. This experimental fact, combined with the assumption, which must be regarded as based upon experiment, that certain types of charge distribution exist, that in fact charge is only to be found in the places where we choose to specify it in any given problem, these are the only facts relying upon experiment for a field whose curl is everywhere zero, and in which the charge density ρ is defined as *Div E*. The law of inverse squares is for this case a result of pure mathematics.[1]

Dielectrics.—As regards the role played by dielectric phenomena in electrostatics little is to be said. The fundamental assumption is that the dielectric may be represented as regards its external action by replacing it by a distribution of polarization P. This turns out to be the mathematical equivalent of a fictitious volume distribution of charge with density $-Div\ P$, and a fictitious surface distribution which at each point of the surface has a density equal to P_n, where the subscript n is to be taken as denoting the component in the direction of the outward normal. Thus, in order to include dielectrics, no further extension is needed in the experimental law as formulated above with the electric field at an external point occupying the position of a subordinate quantity introduced after the truth of the law has been recognized. As regards the field at a point inside the dielectric, this is defined as the quantity obtained by drawing a small cavity about the point, measuring the field therein, and subtracting the portion of the field due to the fictitious charge on the wall of the cavity, this latter being the only part of the field within the cavity which does not attain a definite limit as the size of the cavity is made infinitesimal. We may parenthetically remark that, although it is not customary to do so, we may avoid completely the mention of a cavity by defining the field at a point within the dielectric as the actual field which would be measured at that point if we were to remove the dielectric, and replace it by its fictitious volume and surface charges. The smoothing out process implied in the specification of the fictitious charges carries with it the requirements necessary for insuring a definite limiting value for the field at the point, without resort to the use of a cavity. This method suffers in no respect in comparison with that invoking the use of a cavity in that it requires specification of the fictitious charges before the field can be obtained;

[1] As regards conservation of charge, perfect generality of view-point would hardly call for a more restrictive hypothesis than the assumption of a constant and zero value for the integral of *Div E* for the universe as a whole. Denial of the possibility of coalescence of the ultimate positive and negative units would permit of a more restrictive hypothesis in the form that the volume integrals of *Div E* taken separately for positive and negative values of *Div E*, and in a true differential as distinct from a macroscopic sense, are each individually constant for all time, and are equal, though opposite in sign.

for, although it would be possible to measure the field inside a cavity without invoking the knowledge of the fictitious charges, we should require to know these charges in order to calculate the contribution of that portion of them associated with the wall of the cavity. There is one point in favor of the cavity-definition however. We may construct a long narrow cavity, and can conceivably turn it about within the medium until we find experimentally that the field inside it is parallel to its sides. We may then define the field at a point in the dielectric (if non-crystalline) as the field at a point so chosen within this cavity that the distance of the point from the ends is large compared with the width of the cavity. Such a definition is independent of any theory of fictitious charges or of doublet action of the dielectric. Until we assume some such theory, however, we are unable to attach any properties to the said field. It derives its properties only through the particular theory of the mechanism of the process which has suggested its definition. The most important property of E is that $E+P$ is a solenoidal vector in places where the true macroscopic charge density is zero. The introduction of this vector $D=E+P$ is a mathematical device designed to relieve the mind of the necessity of thinking of the fictitious charges and to allow it to concentrate attention only upon that portion of the total charge which is not directly attributable to the doublets, and whose amount is measured by $Div\ \ D$.

The relation between D and E is a matter of experiment, but it is something which does not partake of the nature of a general law. It is a happy accident that the ratio D/E is a constant in most substances, but it is not a result which may be dignified by being recognized as a fundamental law of physics.

Regarding the boundary conditions at the surface of dielectrics, the continuity of the normal induction is a result following directly from the definition of that quantity; and while the equality of the external and internal tangential intensity is frequently proved by appeals to the conservation of energy and founded on the carrying of a unit charge around a circuit partly in one medium and partly in the other, a close examination of the matter will show that this demonstration assumes the result to be proved, and that the proof itself rests upon the definite assumption as to how the dielectric acts in being the equivalent of fictitious volume and surface charges. This assumption carries with it the equality of the tangential components in question. Any appeal to the conservation of energy as an experimental basis requires the assumption of continuity of potential at the boundary, or at any rate a discontinuity which remains constant to the second order over a first order distance parallel to the boundary. When this assumption is made, however, the equality of the tangential components follows immediately without any appeal to the conservation of energy.

Steady Currents in Continuous Circuits

The unit pole is an obviously obnoxious concept for the purposes of a fundamental definition since it does not exist. The objection may be in part removed by utilizing a unit magnet and defining the field in terms of the couple on it. Even here, a great deal of care is necessary in preserving logical exactness in the definition of magnetic field. (See appendix, Note 1). Two laws are customarily associated with the action of steady currents. The first of these concerns the field due to a current, while the second concerns the forces on a conductor carrying a current. The first law may be expressed in either of two equivalent forms;

> (a) That the field due to a current circuit is the equivalent of that of a magnetic shell of uniform strength whose boundary is coincident with the circuit.
>
> (b) That a current circuit acts as though each element produces a field at a point P in a direction perpendicular to the plane containing the element and radius vector joining it to P, inversely proportional to the square of the distance, and directly proportional to the resolved component of the element perpendicular to the radius vector, and to the current. } B

We have a situation analogous to that met with in the case of electrostatics. If we regard currents as things produced in certain specified ways, and if when we have a number of visible wire circuits we decide to say that what we shall call current exists only in those circuits, and that there are no invisible current circuits in space, then the above law must be regarded as an experimental fact. If, however, anticipating the future needs applicable to the case where we have to deal with invisible movements of electricity, anchor rings of moving electricity in atoms, etc., we define the current density as *curl H*, then remembering that any vector H may be expressed as

$$4\pi H = -Grad\int\int\int\frac{Div\ H}{r}\,d\tau + Curl\int\int\int\frac{Curl\ H}{r}\,d\tau \qquad (5)$$

we see that nothing more than the assumption, or if we like the experimental criterion, that *Div H* is zero is necessary in order that our definition of the current density j as *curl H* shall lead to

$$4\pi H = Curl\int\int\int\frac{j}{r}\,d\tau \qquad (6)$$

which, by simple analytical procedure turns out to be the law of the magnetic shell in the case of a circuit.

Just as in the corresponding case in electrostatics, with ρ defined as *Div E*, the complete role of experiment was played in saying that certain types of charge distribution existed, that in the case of a certain set of bodies there was no charge save on the bodies, so here the complete role of experiment is played in assuring us that when we do to wire circuits those things which we associate with the production of currents in them, the currents will exist in those wires and nowhere else.

The second law concerned with the action of continuous currents has to do with the action of a magnetic field upon a current, and it is customary to derive it as a consequence of a rather unsatisfactory procedure founded upon the consideration of the force due to the current on a pole, and the assignment of forces on the current due to the pole in such a way that the two shall be in statical equilibrium. The force on the current element is then expressed in terms of the field of the pole and the result is extended to apply to the general type of field which may be considered as made up of the field due to a suitable assemblage of poles.

When all is said and done, the law as finally expressed takes the form that,

> The forces and couples on the circuit as a whole are the same as would be secured were we to assign to each element ds, of the current circuit, a force of the form
>
> $$\Delta F = ds[i.H] \tag{7}$$

Now just as in the electrostatical case the introduction of the field E, with ρ defined as *Div* E, necessitated the assumption of a law as to the types of charge distribution which are possible, and then another law stating how the field acts upon the charged bodies, while unless we are interested in the value of the field for its own sake, its introduction is unnecessary, so here, unless we are interested as a matter of pure philosophical speculation in the question of the magnetic field itself, we may relegate it to a subsidiary role, formulate the law which is to be considered as the result of experiment without it, and then introduce it again, defined in a new way, however, for the purposes of mathematical convenience.

We may in fact state in the following way the complete law of the action for continuous steady currents, i. e., the only law which calls for experimental investigation.

> Suppose we have a number of material current circuits. Then it is possible to assign to the circuits numbers i_1, i_2, i_3, etc., such that if H be *defined* as
>
> $$H = \Sigma curl \int \frac{i}{r} ds \tag{8}$$
>
> the forces and couples between the current circuits are given by assigning to each element a force
>
> $$\Delta F = [i.H] ds \tag{9}$$
>
> where at each point of the nth circuit, i is to be regarded as a vector of modulus i_n. Electrostatic forces are supposed eliminated by screens, which prevent no difficulties for steady currents.

C

Here H is retained merely as a symbol for mathematical convenience. It involves nothing about a pole or a magnet. Indeed, we could dispense with it altogether by expressing our force on the element ds_n as,

$$\Delta F_n = \left[i_n \Sigma curl \int \frac{i}{r} ds \right] ds_n \tag{10}$$

It will be observed that there is a certain analogy in language between the expression of this law and the fundamental law of electrostatics as given on page 6. In the present case we have omitted the part which would correspond to the requirements of constancy of total charge in electrostatics. Were we to incorporate this statement in the present case it would necessitate the addition to C of a statement to the effect that,

> **If the circuits be then moved to other positions a similar law holds, and the new values of the i's for the several circuits will be the same as the old values.** } D

The objection to this addition is that, in the ideal and general case, it would not be true, as will immediately be surmised if one thinks of the permanent alteration in current which would be produced in a resistanceless circuit by electromagnetic induction if its position were changed in such a way as to alter the magnetic flux through it. In all practical cases, however, where we have resistance, we know that such effects as those cited would be only transitory, the currents produced by electromagnetic induction dying to negligible values in sufficient time, leaving only the contributions to the currents by those other sources, batteries, etc., with which we are principally concerned in speaking of steady currents. If, however, we wish to retain a degree of formality which is perhaps out of proportion to its practical usefulness we should limit ourselves to the statement that "there are cases in which the law stated in C with the addition D is true."

It must be pointed out that, even though we restrict ourselves to the form C of the law without the addition of D, the law itself forms more than a mere definition of the current strength. For while, with n given circuits, we could write down n equations for the force components on the circuits in any direction, and determine values of i so as to satisfy these relations, independently of the form of the law which had been assumed, and while nothing more than the vectorial form of the relations would be necessary in order that these same i's would lead to the proper components of the force in the perpendicular directions, it would not necessarily follow that the couples would be correctly given by the same i's as gave the proper forces. It may be argued that, in the practical operation of testing the law of action of currents, we do not go through the artificial process of assigning numbers i_1, i_2, etc., but adopt processes which are much more in line with our physical conceptions. A little thought will show that if we remove from our minds all pictures concerned with the visualization of the process, and concentrate only upon what we actually do, our tests do really take this form. Thus, to take the simplest possible case, that of two circuits with constant currents, were we to set out to test our law, we should write down an equation founded upon equation (10), for the couple on one of the circuits and we should calculate two numbers i_1 and i_2, one for each circuit. We should then move the circuits to different relative positions and test whether the same two numbers i_1 and i_2 would serve in conjunction with equation (10) to give the couple in this case.

Although in the form of expression of the general law as symbolized by (10) the magnetic field has been dispensed with, it is naturally a convenience for mathematical manipulation to reintroduce it defined by equation (8). The properties of H are now of course all contained in its definition, and this definition gives to it the property of being expressible in terms of magnetic shells as before. Even the relation $Div\ H=0$ is secured by the relation (8), since the divergence of a "curl" is necessarily zero.

In turning things around in this way and formulating everything in terms of one experimental statement we may have a suspicion that we have lost something, and indeed we have, but happily something of very little value. In terms of the present specification there is nothing to say how that specialized piece of steel we call a magnet will behave under the influence of the currents and the field H as now defined. If we wish to bring the steel magnet in as a matter of historical interest, we may do so by adding an additional experimental conclusion, viz., the conclusion that the said piece of steel can be treated as regards its behavior as though made up of current whirls.

In concluding these statements regarding steady currents we may remark that in the formulation and definition of the field H in terms of the currents, and of the latter in terms of the forces between the circuits, we formulate our laws according to the actual principles which are used in accurately measuring the currents and fields.

It must be added that the proportionality between the currents as above defined and the currents as defined by electrolysis is to be regarded as an extra fact which must seek its foundation in experiment.

Varying Currents in Closed Circuits

Dynamical systems.—Since so much of the formulation of the general laws of electromagnetism is bound up with dynamical methods, the deduction of laws from dynamical principles and so forth, one has to consider to what extent these processes constitute proofs of the results deduced, and in what place experiment plays its part. It will be well, therefore, to make a few remarks about dynamical systems and developments in general. First, what is a dynamical system? This is of course a matter of definition. There are several definitions, but the one on which it will serve to fix attention is this:

Suppose it is possible to define a set of measurable quantities θ_1, θ_2, θ_3, etc., and to build up a homogeneous quadratic function T of the time rates of change of these quantities, and another function V of the quantities themselves such that the motions of the system are found to be in agreement with the equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} = -\frac{\partial V}{\partial \theta_1};\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} = -\frac{\partial V}{\partial \theta_2};\text{etc.,} \tag{11}$$

then the system is said to be a dynamical system. If it be more convenient, we may use in place of (11), their analytical equivalent as specified by the Hamiltonian principle.

$$\delta \int (T-V)\,dt = 0 \tag{12}$$

It is occasionally necessary to subject the coordinates to constraints, with the introduction of undetermined multipliers in a perfectly well known manner which we need not belabor.

The only birth-mark of this rather artificial specification of a dynamical system which gives it, in its general form, some connection with those particular systems of particles which are discussed in text books in which Lagrange's equations are deduced, the only birth-mark which relates it to what, in our youth, we regarded as familiar, is a property of such a system which was discussed by Poincaré, to the effect that whatever quantity T we assign of the above type, and whatever quantity V we assign of the above type, it is always possible to assign a system of particles with constant m's for which $\Sigma\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ is this function T, and to assign central forces between the particles, and on them from without, such that, when we go through the process of building up Lagrange's equations for these particles, our V will turn out to be the quantity whose derivatives are these external forces; in fact, it is always possible to assign a system of particles controlled by central forces such that the Lagrangian equations which we have are the generalized equations for these particles. Moreover, this can be done in an infinite number of ways. We must not always expect that the particles to which the dynamical system is to be referred in this way are necessarily such particles as electrons for example. There is danger that when we meet things like electrons, we shall at once assume that these are the fundamental particles in terms of which the system is to be expressed if it is to figure as a dynamical system. There is danger of our being led, as some have been led, to the conclusion that, because we cannot express the behavior of the electromagnetic field in a dynamical form with part of the kinetic energy T given by $\Sigma\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ for the electrons, therefore the system is not expressible in dynamical form, whereas the very nature of the dynamical development so expresses it. There is indeed no earthly reason why the ultimate particles to which the system might be referred should have any simple relation whatever to such particles, electrons, and so forth, as we happen to find flying about.

With regard again to the role of theory versus experiment, it must be remembered that the function of a dynamical development in electrodynamics is not to prove anything. The situation is not one where we ascertain in some mysterious way, as a result of crude experimental illustration or otherwise, that certain expressions for T and V are the right ones and then go ahead and prove something which calls for no further experimental investigation if the laws of dynamics are true. The situation is rather the reverse. The things to be experimentally verified are the results, and what has been done is to mould the whole scheme into a dynamical form by finding what expressions for T and V must be chosen in order that the application of the equations of Lagrange shall lead to these experimentally established results. The fact that the expressions chosen for T and V fill this role is the only criterion for their truth. Any one who adopts any other view, and tries, for example, to argue that $\frac{1}{2}E^2$ represents the potential energy per unit volume in the general case, because this expression was deduced in the case of electrostatics, will find a very confusing piece of work on his hands if he attempts to carry out, for the general case, any argument analogous to that which he employed as a proof in the statical case.

In Maxwell's original investigation applicable to continuous circuits, the starting point was the assumption that the kinetic energy was a sum of two parts, one a material part having exactly the form applicable in the absence of currents and another part in which the velocities were represented by the currents, this part being in fact a homogeneous quadratic function of the currents of the form

$$T=\frac{1}{2}L_{11}\,i_1^2+L_{12}i_1i_2+\;.\;.\;.\;L_{mn}\,i_mi_n\;.\;.\;.\qquad(13)$$

with the L's functions only of the material coordinates. The absence of product terms of material and electrical velocities was inferred from the fact that the presence of terms of this type in the energy would cause Lagrange's equations to predict phenomena which were found experimentally not to exist. Terms of this kind would, for example, lead to the conclusion that there would be mechanical forces on the conductors depending upon the rate of change of the currents, whereas experiment shows that all such forces depend only on the currents themselves. Then again there would arise mechanical forces on conductors of amounts proportional to the product of the mechanical velocities and the currents, forces which were again sought without success. Again there would be electromotive forces as a result of mere acceleration of the conductors even though initially they had no currents in them. It may be remarked that, from the standpoint of electron theory these effects sought for by Maxwell should actually exist. They should be just within the range of measurement, and indeed, some of them have been observed, although of course they were quite beyond the means of detection at Maxwell's disposal.

The utilization of the foregoing energy expressions in the Lagrangian equations leads immediately to the following two expressions for the generalized mechanical force X on a circuit denoted by subscript (1), and the generalized electrical force Y operating on that circuit. These generalized forces are defined as the coefficients of the displacement of the corresponding coordinate in the expression for the virtual work.

$$X_1 = \dot{i}_2\dot{i}_1 \frac{dL_{12}}{dx_1} + \dot{i}_3\dot{i}_1 \frac{dL_{13}}{dx_1} + \dots \qquad (14)$$

$$Y_1 = \frac{d}{dt}\left(L_{11}\dot{i}_1 + L_{12}\dot{i}_2 + L_{13}\dot{i}_3 + \dots\dots\right) \qquad (15)$$

These then are the expressions which it remains for experiment to test if the above expression for the energy is to be regarded as a possible one to figure in a dynamical specification of the laws.

It is a matter of pure analysis to show that provided that L_{mn} be chosen as the purely geometrical function of the shape and positions of the circuits given by

$$L_{mn} = \int\int \frac{\cos \varphi \, ds_1 ds_2}{r} \qquad (16)$$

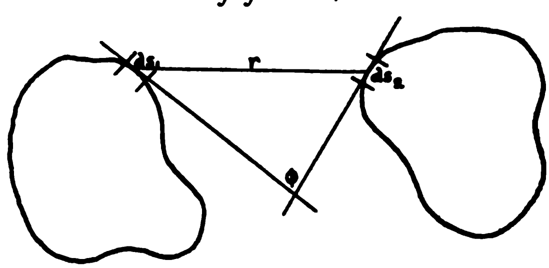

FIG. 1.

expression (14) is the analytical equivalent of equation (10) which we decided to regard as the experimental foundation for the laws for steady currents. The correlation for the case of steady currents fixes the L's to have the values given by (16), but once this is done, the expression

(14) reverts to that form for steady currents, and retains this form as regards what it has to say about variable currents.

Moreover, it follows as a pure analytical consequence of the choice of the above expression for the L's that the expression in parentheses in equation (15) represents the flux through the circuit (1) of the quantity H already defined by (8), so that (15) bears a close resemblance to Faraday's law.

As regards the mechanical action of the current circuits on each other, the law to be tested is then the same as the law for steady currents. The question is whether it is possible to chose numbers i_1, i_2, etc., one for each circuit, such that the law of mechanical action may be expressed in terms of them in the manner we have specified, after inclusion of appropriate electrostatic forces, since shielding is not now permissible. Although it would be very difficult to show by ordinary laboratory experiments that such was impossible, we may draw upon our further knowledge of the laws for open circuits to convince ourselves that were we able to carry out the experiment with sufficient precision, it would be found impossible to choose numbers i_1, i_2, etc., fulfilling this condition, even by inclusion of electrostatic forces. The defect of the theory is that it ignores the finite time of propagation of effects. In the open circuit theory where, for the first time, the currents of which we have spoken become definitely associated with electric charges, it becomes necessary in order to retain certain properties which are at the basis of our physical concepts of the subject, to introduce the displacement current density $\dot{E}$, which, of course, immediately introduces the analytical characteristics of a finite rate of propogation of effects. Continuous current theory which had not yet recognized any necessary connections between the currents as defined and the motions of electric charges would not be impelled on its own account by any strong logical call for the existence of this displacement current. Once $\dot{E}$ has, however, been introduced in the more general case it cannot be summarily discarded for the more special case unless it disappears of its own accord, which it does not do.

We may infer from this that although it is the more general case for open circuits which has suggested the introduction of the displacement current, the field of closed currents would not, at any rate from an ideal standpoint, have to rely upon open circuits to show the incorrectness of the theory we have discussed, but would, of its own accord, reveal this incorrectness in the law of mechanical action of the circuits if only the experiments could be carried out with sufficient refinement.

As regards equation (15) giving the electromotive force, this would be Faraday's law provided that a way could be found of associating Y with the electromotive force in the sense that that term is used in Faraday's law. It is important to realize that, at this stage no suggestion has been made as to Y_1's having anything to do with electricity. It is defined as the coefficient of the current i_1 in the expression for the rate of doing work by the external forces. As such, the nearest we could come to its experimental realization would be to measure it as the coefficient of the current i_1 in the expression for the rate of generation of heat in the circuit (1), a procedure which would, however, in order to maintain logical precision, necessitate the introduction of a number of other postulates and assumptions, or experimental facts, having but a limited bearing upon the matter in hand.

We might take (15) as the definition of electromotive force, and then define resistance as the ratio of the e.m.f. so defined to the current in the conductor (1). The sole fact which would then come out of experiment would be that there exist conductors for which the resistance so defined is independent of the current (i. e., Ohm's law holds). This is really all that is proved by an experiment in which the throw of a ballistic galvanometer is measured as the result of starting a current in some other circuit, for Ohm's law is assumed in the interpretation of the result in relation to what it is desired to prove.

One might say, why not introduce physical reality into the quantity Y, and at the same time tie it up with electrostatic phenomena by introducing a battery,[1] which is a contrivance having a very obvious connection with electrostatics, in view of the fact that its plates are at different potentials as tested electrostatically. The trouble is that a battery is not a thing which it is very easy to define mathematically. It is a sort of electrostatic device which produces a closed current; and to account for this astonishing performance, we have to introduce a lot of extraneous experimental and theoretical material, such as Nernst solution pressures, etc., with which it is undesirable to encumber the argument for the sake of forcing the connection between currents and electrostatics into the discussion at this stage, particularly as the connection becomes more naturally discussed in the more general analysis for open circuits, and as the whole theory in the form in which we are discussing it is experimentally wrong, since it neglects the finite time of propagation of effects.

One might say, why discuss a theory which is wrong? The answer is, that for many practical purposes it is sufficiently nearly right, and is not in itself logically inconsistent in its individual parts. All that it is useful to assert, as regards the Faraday law, at this stage of the development, is that equation (15) is not to be regarded as an experimental law, but as the definition of e.m.f. The ratio of the so defined e.m.f. to the current in the wire is called the resistance in electromagnetic

[1] A condenser is not allowable in the present stage of our discussion, since we are here confining ourselves to the laws of closed circuits.

measure, and the usefulness of this definition lies in the fact that, for many substances, if the changes in current are not too rapid, the so defined resistance is a constant independent of the magnitude of the current. Connections with electrostatics can only be brought in with the accompaniment of extraneous and cumbersome considerations involving other matters, and are better relegated to the more general part of the subject concerned with open circuits.

Without incorporating the statement as part of any general law, however, it may be convenient to recognize, as an experimental fact, that the quantity defined as the electromotive force in (15) is approximately proportional at any instant to the open circuit difference of electrostatic potential between the poles of a battery which, when forming part of the circuit, would at that instant result in zero current.

THE GENERAL EQUATIONS

The equations are usually written:

$$\frac{1}{c}\left(\rho v+\frac{\partial E}{\partial t}\right)=Curl\ H \tag{17}$$

$$-\frac{1}{c}\frac{\partial H}{\partial t}=Curl\ E \tag{18}$$

$$Div\ E=\rho \tag{19}$$

$$Div\ H=0 \tag{20}$$

and to these we must add the force equation, which states that,
"The motion of an element of charge (or in the form in which the law is used, an electron) is determined by the condition that the value of

$$\rho\{E+[vH]/c\} \tag{21}$$

integrated over it is zero."

It will perhaps conduce to clarity if we pause for a moment to consider the manner in which these equations are related to each other as part of a dynamical system.

The expression

$$T=\tfrac{1}{2}L_{11}\,i_1^2+L_{12}\,i_1i_2\quad+\text{etc.} \tag{22}$$

taken by Maxwell for the kinetic energy of a system of current circuits becomes generalized into the form

$$T=\tfrac{1}{2}\int\int\int(J_xj_x+J_yj_y+J_zj_z)d\tau \tag{23}$$

when applied to continuous distributions of current with density j. In this expression,

$$4\pi J=\int\int\int\frac{j}{r}\,d\tau \tag{24}$$

In writing (22) in the form (23) it is implicitly assumed that

$$Div\, j = O \tag{25}$$

Now as a result of experiments indicating that moving charges in the statical sense produce effects similar to what we have regarded as electric currents, we conceive the likelihood of its being possible to express the currents in some manner which associates them with moving charges. What is next done is to replace the current j in expressions (23) and (24) by $1/c$ times $(\rho v + \dot{E})$, where c is at present a constant to be subsequently determined by experiment, and try whether an expression of this form for T, and a suitably chosen V can be made to serve as kinetic and potential energies which, by the application of Hamilton's principle, will lead to equations which represent experimental facts. The addition of the term $\dot{E}$, is more or less arbitrary, and is determined by the fact that anything which has to take the place of j and retain the feature which is characteristic of that quantity in continuous current theory, the feature of having zero divergence, must itself have zero divergence. To restiict ρv to have zero divergence would be to place a greater restriction on our analysis than we believe would be warranted by natuıe. For if v is the velocity of the charge density in the sense that, in a particle of finite size, the value of v at a plane multiplied by the density there represents the rate of disappearance of charge from one side, and if we are to retain the idea of indestructibility of charge, we must have the equation of continuity

$$-\frac{\partial \rho}{\partial t} = Div\, \rho v \tag{26}$$

so that, with the definition $\rho = Div\, E$, it is the divergence of $(\rho v + \dot{E})$ which is to be zero, and $(\rho v + \dot{E})$, or something proportional thereto, consequently suggests itself as the quantity to take the place of j in this analysis, although of course, as regards the mere requirement of zero divergence, we might add any other vector whose divergence is zero. The expression which is chosen for the potential energy is

$$V = \frac{1}{2}\int\int\int (E_x^2 + E_y^2 + E_z^2)\, d\tau \tag{27}$$

which is the same expression as is deduced for the case of electrostatics. The best that can be said for this, in the way of philosophical justification, is the rather weak statement that the potential energy ought not to depend upon how the field is produced. Thus, with T defined as

$$T = \frac{1}{2c}\int\int\int \{J_x(\rho v_x + \dot{E}_x) + J_y(\rho v_y + \dot{E}_y) + J_z(\rho v_z + \dot{E}_z)\}\, d\tau \tag{28}$$

where

$$4\pi J=\frac{1}{c}\int\int\int\frac{(\rho v+\dot{E})}{r}\,d\tau \tag{29}$$

and with V defined as above, and $Div\ J=O$, the Hamiltonian principle is applied with E and the displacement of the charge as coordinates connected by the restricting relation

$$Div\ E=\rho$$

with the result that equations (18) and (21) are obtained with H defined as

$$H=Curl\ J$$

which, since $Div\ J$ is zero is the equivalent of equation (17), thus showing this equation to be the equivalent of a definition of H.

The truth of whatever experimental facts are implied in equations (17) to (21) is of course the sole justification for the choice of the foregoing expressions for T and V.

Concerning other dynamical developments, reference may be made to the appendix to this paper, Note 2.

The significance of the equations and of the quantities occuring in them.—In reviewing a set of relations such as those under discussion,the question of which quantities are defined independently of the equations in which they occur and which are merely defined by those equations is one of considerable pertinence; and, the answer to the question may, even in a given set of relations, be different according to the point of view which is adopted as to the interpretation of relations. A definition of a quantity should be a statement of the sense in which it is to be used; and, it is desirable, if possible, that it contain a specification of the manner in which the quantity should be measured. In any case, even though infinite experimental skill be called for in the realization of the measure of a quantity in the strict sense of its definition, that definition should not call for the realization of conditions which are logically inconsistent with the laws of the subject which are being discussed. Thus, to define the magnetic field at a point inside an electron in terms of the couple on a magnet placed at the point is not merely to define it in a manner incapable of experimental realization on account of difficulty, but in a way which is logically meaningless. Moreover, in so far as the use of the equations, in their relation to the correlation of what is actually observed in experiments, never requires that we attempt to put a magnet inside an electron for the purpose of obtaining the data which are the ultimate goal of the investigation, it should certainly be possible to formulate what the equations have to say without introducing this meaningless concept.

In the process of tracing the analogy of our equations with those of some dynamical system, definitions do not raise such difficult questions, at least in respect of the dynamical arguments themselves. The purpose of the dynamical argument is to build a model satisfying the equations, and in doing this, we may define things like E and H in any way we happen to find convenient. Thus, to take an illustration which is purposely chosen in an artificial manner in order to bring out the point, let us suppose all space filled with particles, and let us denote by E the displacement of a particle of type A from its initial position as determined in some frame of reference. Let there be in addition other particles of type B, whose number per c.c. is denoted by ρ. Let there be some sort of binding between the particles so that the relation $Div\ E=\rho$ is always satisfied. Let v be the velocity of the particles of the type B as measured macroscopically in an element of volume. Then, if we choose quantities T and V as in (27) and (28), and develop the analysis in the manner there sketched, we shall of course arrive at the relations (17)–(21), with H defined as $Curl\ J$. It is not implied that the model arrived at is in any sense a true picture of the processes going on in the aether. The use of the particles in our thoughts about the model in no sense implies that there are, in the aether, particles of which E represents the displacement, and much less is it implied that the particles of which we have spoken are those fundamental but hypothetical particles to which, according to Poincaré, all dynamical systems may be referred. In fact, as already enforced, the dynamical argument does not constitute a proof of the relations, nor does it pretend to a unique discovery of an actual mechanism. The ultimate criteria for the truth of the relations must rest upon the experimental verification of such facts as are implied in the relations themselves, and the definitions of the quantities occurring in the relations need not, and in general can not, follow the definitions which were convenient for the dynamical analysis. Thus, it would be perfectly meaningless to define E as the displacement which a particle of the aether would suffer if the aether were composed of particles. Such a definition would moreover be useless in the sense that it would not be of a kind which would permit realization to the extent of allowing one to test the validity of any law in which the so-defined quantity occurred.

Another aspect of the difference in point of view appropriate to one who considers the dynamical development and one who considers the equations on their own merits is illustrated by the relation $\rho=Div\ E$. In the dynamical development, this was the condition of constraint introduced between two types of coordinates representing respectively the displacements of the particles of the A and B types. It was part of the law that these two things should be bound by a restricting relation.

From the point of view of one who considers only the final equations, however, the situation is slightly otherwise. If the nature of his measurements are such that he has, in terms of them, a perfectly definite definition of E, and if, in the process of calculating ρ, he always does so by the relation $\rho = Div\ E$, this relation becomes the definition of ρ. On the other hand, if he has some independent way of measuring ρ, and E, such a relation becomes part of the law. Again, if the experimenter decides upon some usable definitions for ρ and v, it may be that the equations (17)–(20) form largely a definition of the remainder of the quantities involved. It becomes immediately a question of interest as to how many of the quantities must be independently defined in order that the equations shall contain the definitions of the remainder. We shall presently probe this matter further in an endeavour to ascertain the most logical starting point for our definitions in order that those definitions shall represent the actual sense in which we use the quantities in our calculations. For the moment, however, it will be of interest to trace the meaning of the equations in the light of one who takes E and H as his fundamental starting points.

In the first place, (19) must be regarded as the definition of ρ, which, without such definition, would be meaningless, particularly in the case of moving charges. Equation (20) is merely a statement of the restriction of our interest to those problems in which this relation holds, and is not contained in the other equations as they stand. Put physically it is the analytical statement that we believe that no magnetic poles occur in nature.

From the stand point of one who takes E and H as his fundamentals, it can be shown that equation (17) is largely a definition of v, which, in itself, and without specific definition, would have no real meaning inside an electron for example. For further extension of this idea see appendix, Note 3.

It is important to realize that equations 17-20 give the fields when the motions of the charges are given, but they do not say anything about how the charges move as a result of the fields in which they exist.

It is convenient to express the solutions of the equations in terms of potentials slightly different from those used in their dynamical derivation. The desired solutions are

$$H = Curl\ U \tag{30}$$

$$E = -\frac{1}{c}\frac{\partial U}{\partial t} - Grad\ \varphi \tag{31}$$

where

$$4\pi\varphi = \iiint \frac{[\rho]}{r}\,d\tau \quad , \quad 4\pi c U = \iiint \frac{[\rho v]}{r}\,d\tau \tag{32}$$

the possible values of the v's to be associated with the ρ's being restricted by the condition

$$-\frac{1}{c}\frac{\partial \varphi}{\partial t}=Div\ U. \tag{33}$$

The square brackets are to be taken as indicating that the potentials are to be calculated in the usual retarded sense. They are the symbols of what may be interpreted physically as a finite rate of propogation of effects.

The vital importance of the term $\dot{E}$ in (17) is illustrated by the fact that, if it be omitted, expressions for E and H are obtained exactly like those given by equations (30) to (33), but with the retarded feature as symbolized in the square brackets absent. In other words, the physical language appropriate to that case is the language of instantaneous propagation.

Happily the equations representing φ and U and consequently E and H are made up of individual contributions from the volume elements in space. It is therefore possible to attach a meaning to the idea of the field due to one piece of electricity as distinct from that due to another. This is of course purely a matter of definition in the sense, for example, that if we have two charges A and B in each other's vicinity, and wish to speak of the field at a point P due to A alone, we decide to mean by this the portions of the total E and H which are contributed by those portions of φ and U which are contributed by the charge A alone.

Attention must be called to one further point in these solutions. Equations (30) to (32) enable us to calculate the fields provided that we assign the motions of the charges, and we could calculate from them fields whatever motions we assigned. Equation (17), however, contains the equation of continuity, and forbids our assigning motions which would in themselves violate this condition. They forbid, for example, that we set a problem in which v is everywhere zero, and ρ gradually diminishes to zero from a finite value. Equation (33) is the analytical guardian of the equation of continuity in forbidding us to assign in (32), values of ρ and ρv which would violate it. The equation of continuity does not of course follow as an absolute logical necessity without need of analytical specification. It does not apply to everything. It does not apply to the people in this country. The increase of population per year is not equal to the difference between the number who come into the country and the number who go out in that time, for some are born here, and some die here. Electricity might die, and it requires analytical specification to keep it alive.

We have remarked that equations 17–20, or their solutions, give the fields when the motion of the charges are given, but that they do not give the motions of the charges themselves in terms of the fields. This is the duty of the force equation. It would seem that, in accordance with the manner in which this law comes out of the dynamical analysis, it states that the electricity must move in such a way that the total force as measured by $E + [vH]/c$ must be zero for each point. In the form in which the result is used, however, in its application to an electron, it takes the less restricted form that the electron moves in such a way that the value of ρ times this vector integrates to zero when evaluated for the whole of it. Calling this vector a force, which name in itself does not of course imply anything about it to justify surprise in case the language used in connection with the said force should sound strange if considered in relation to ordinary mechanics, our law assumes the form that the electron moves in such a way that the total force which it exerts on itself due to its motion is equal and opposite to the force on it due to the external field. In other words, the law assumes the form

$$\iiint\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau+\iiint\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau=0 \qquad (34)$$

where E_1 and H_1 refer to the field produced by the electron itself at points inside it, and E_2 and H_2 refer to the field due to external causes. Before the force which the electron exerts upon itself, as a result of its motion, can be calculated, it is necessary to make a hypothesis as to what an electron is, as to how ρ is distributed within it, and as to how its shape and charge distribution change with the velocity. In the theory itself there is nothing to indicate this. Before pursuing this matter any further in a manner implying the existence of electrons, it will be well to consider the status which the equations attain immediately we specify any law of motion, i.e., any law stating a relation between E and H at a point, and the velocity, acceleration, rate of change of acceleration etc. at that point.

Foundation of electrodynamic laws in relation to actual observation.— Suppose in fact we write down some such equation as

$$\psi(\rho, v, \dot{v}, \ddot{v}, \text{etc.})=E+\frac{[vH]}{c} \qquad (35)$$

where the function ψ is supposed to be specified, and where we have not written the right hand side in the form shown as a result of any general necessity for this form involved in what we wish to say, but simply in order to use in the illustration something which is not unnecessarily artificial and far removed from the facts as they turn out in practice. Directly we have some such equation as (35), we are in a position to formulate in one statement all that electromagnetic theory needs as verification from experiment in order that everything else may follow as a mere matter of definition.

Remembering that nobody can see a charge or measure its velocity with a meter stick, it will be well to avoid limiting the statement of the experimental law by the artificial conception that these things can be measured. We shall consequently formulate the law in such a way as to avoid these objections. We must suppose that we have decided upon some equation such as (35) which we are going to assert that experiment will justify. Let us write this equation down, replacing E and H, however by the values given by (30)–(32) so that the equation assumes the form

$$\psi(\rho, v, \dot{v}, \text{ etc}) = -\frac{1}{4\pi c^2}\frac{\partial}{\partial t}\int\int\int\frac{[\rho v]}{r}\,d\tau$$
$$-Grad\int\int\int\frac{[\rho]}{4\pi r}\,d\tau + \left[\frac{v}{4\pi c^2}\,Curl\int\int\int\frac{[\rho v]}{r}\,d\tau\right] \qquad (36)$$

it being understood that c is a constant number. The large square bracket indicates a vector product, while the small square brackets indicate retarded potentials. Let us also write down the equation of continuity

$$-\frac{\partial\rho}{\partial t} = Div\ \rho v \qquad (37)$$

We may then express the general law as follows:

> It is always possible to find ρ and v as functions of position and time, and solutions of (36) and (37), such that the motion of the boundary of any body, beam of luminosity, etc. of macroscopic size (and which represents what is ultimately observed in the experiment), may be regarded as given by the value of v as obtained for the points of the boundary, this value being, however, smoothed out in the macroscopic sense. In the case of a statical boundary, the position is to be determined either from the time integral of the velocities, or by the criterion that ρ suffers a discontinuity there in the macroscopic sense. F

It must be pointed out that all observable phenomena are included as particular cases of this law. Thus, if a beam of electrons enters a Faraday cylinder, and charges an electrometer, the needle ultimately moves; and, in the full generality of the case, this motion may be described in the above manner. Again, even though the observer should base his conclusions upon measurements of the path of a beam of light produced by the passage of a beam of electrons through an ionized gas, the boundary of this beam is really something whose position may be regarded as specified in the above sense. Again, if one is discussing the amount of scattering of a beam of electrons in passing through a sheet of matter, the movement of the needle of the electrometer is ultimately to be regarded as determined by v at its various parts, this v being bound up with the v's and ρ's in the electrons of the atoms which are causing the scattering, in such a way that the whole set of motions and densities satisfies equations (36) and (37).

Redefinition of the field vectors.—Equation (36) does not contain E or H, but being of a rather cumbersome form, it may be a convenience to introduce certain quantities φ, U, E, H, *defined* as follows:

$$4\pi\varphi=\iiint\frac{[\rho]}{r}d\tau \quad , \quad 4\pi cU=\iiint\frac{[\rho v]}{r}d\tau \tag{38}$$

$$E=-\frac{1}{c}\frac{\partial U}{\partial t}-Grad\ \varphi \tag{39}$$

$$H=Curl\ U \tag{40}$$

Then, in view of the restriction which was specified in (37) and which has controlled the relation between ρ and ρv to some extent, the above definitions alone insure the following relations

$$\frac{1}{c}\left(\rho v+\frac{\partial E}{\partial t}\right)=Curl\ H \tag{41}$$

$$-\frac{1}{c}\frac{\partial H}{\partial t}=Curl\ E \tag{42}$$

$$Div\ E=\rho \tag{43}$$

$$Div\ H=0 \tag{44}$$

It is important to realize, however that E and H have now been relegated to the role of subsidiary quantities introduced for convenience of analysis, but involving nothing about unit pole or unit charge, and the equations connecting them result from their *definitions*, the sole law to be sought from experiment being contained in the statement F made above.

Having reduced E and H to their proper degree of subordination it is not the purpose to discard them, or even to deny ourselves the physical satisfaction of visualizing them in terms of an aether. Whenever they become too assertive, however, by demanding something on their own account, we shall know how to assure them of their proper position in relation to the true representatives of the law, equations (36) and (37).

The value of the field vectors.—In spite of the apparent artificiality of the law as given above, a little consideration will show that a law of this form is what we actually use in all of our calculations. We make use of the properties of substances for the purpose of making short cuts in our calculations, and restricting the generality of our solutions, but would readily be willing to admit that these are only used for convenience and do not figure as parts of the general law. Thus, in the problem of the electrometer and Faraday cylinder cited above, we find it of enormous convenience to recognize that the Faraday cylinder, quadrants of the electrometer, and connecting wires are what we call conductors. But what is a conductor? It is something in which the quantity E defined above is zero for stationary conditions. This criterion enables us very quickly to determine the distribution of ρ over the Faraday cylinder, electrometer quadrants etc., and to be assured of the uniqueness of our solution on the basis of the ordinary uniqueness theorem for electrostatics. Our restriction to electrostatics as regards the Faraday cylinder and electrometer, our belief that those oscillations which were started when the electricity rushed into the quadrants will not persist for a long time, is again something which the general law cannot be expected to tell us, so again we invoke a special property of materials viz. resistance, which insures that, provided we wait long enough, we shall really arrive at a statical distribution in the electrometer. Then, as regards the motion of the needle, in the full generality of the case, this needle would be regarded as distribution of electronic density of a complicated kind. Its very moment of inertia would be sought in the electromagnetic masses of its molecules. The torsion in its fiber would find its representative in electromagnetic phenomena. In fact, the resultant motion of the needle as a whole could really be described in the sense implied in our general law without the invocation of moments of inertia, fiber torsion, etc. Here, however, we make use of our practical knowledge of matters not pertaining to the general law to simplify the procedure. The study of mass conceived as having an electromagnetic origin enables us to surmise that provided the needle does not move so rapidly that the velocity of its parts become comparable with that of light, its angular acceleration is proportional to the couple on it as determined from the electrical forces in the sense ordinarily understood[1], in combination with the torsion of

[1] It must be pointed out that for the case where the equation of the type (35) is a special case of (34), and in view of the definitions of E and H given in (38)–(42), there is an ambiguity as regards the value of ρ at a point in any given problem. In fact, an increase of ρ in the same proportion throughout will not alter matters as concerns the solution sought. If, however, in examining the motion of any of the material bodies which happens to move with small velocity, we calculate the mechanical forces on it from the accelerations, we have a means of fixing the scale of magnitude of ρ by making the electrical force equal to the mechanical. It is of course only for the purpose of correlating the electrical forces with mechanical forces as ordinarily defined that it becomes necessary to fix the scale of ρ at all. Once the scale of ρ is fixed it is but a short step to the realization of the ordinarily defined unit of charge. When this is done, it is only a small step to show that c represents the ratio of the electromagnetic unit of current as defined in terms of the forces between currents to the electrostatic unit as defined in the manner cited above. As a matter of fact it is not even necessary to definitely fix the meaning of the electrostatic unit before showing that c represents the ratio of the units, whatever be the choice for the absolute magnitude of the electrostatic unit.

the fiber as measured mechanically. It is in order to specify these characteristic restrictions which are peculiar to matter, but are not part of any general law, that the introduction of the quantities E and H finds one of its greatest fields of usefulness, for it is in terms of them that these restrictions become expressed. While it is convenient and permissible to use these subsidiary properties of conductivity, etc., it must be clearly recognized that their use is only justified in so far as they are subservient to the general law of the type (36)-(37). The ultimate solution may satisfy certain special conditions arising from the existence of conductivity, etc., in the parts of the apparatus, but as far as a general law of nature is concerned the essential thing is that it shall conform to some such conditions as (36)-(37).

In case any one should demur against the complete relegation of E and H to the role of subsidiary quantities defined by equations (38)–(40), we may point out that it is only in the sense of such a definition that we actually *use* them in many of the problems which are of most importance to us. Thus, suppose we should wish to determine by experiment how the field E varies with position within an electron of the atom of a certain metal, how should we do it? We should, certainly, not attempt to put a unit charge there. We should make up a hypothesis as to the distributions of the electrons and of their velocities and of the values of ρ and v within the electrons. We should write down the relations of the type (36) and (37) which we accepted as the true ones, using possibly the subsidiary scaffolding (17) to (21) for mathematical convenience, and we should try, by means of the distribution of ρ and v which we had assumed, to predict some macroscopically observable phenomena in terms of which we could make our test. We should adjust our theory of the ρ's and v's until we did secure a system which represented all the facts which we could observe, and having found this theory, we should proceed to calculate our fields inside the electron in terms of the definitions given by equations (38)–(40). It is true that our solutions for ρ and v might not be unique as regards their power to represent the observable phenomena; but, they would be unique in the practical sense that they agreed with our observations as far as we were able to make the test.

It may be urged that we may determine the field inside an atom, for example, by shooting an electron through it and calculating the deviation in its path. The difficulty arising here is concerned with the fact that the field which is defined in terms of the motion of the electron as a whole, is too coarse-grained a quantity to serve as applicable to a point inside the electron. Nevertheless, there are certain interesting aspects of the definition of a field in terms of the motion of an electron as a whole, for which reference may be made to Appendix, Note 4.

The degree of fundamentality of the circuital relations.—In line with the above standpoint as to the subordinate positions of E and H, and of the circuital relations themselves, since these are determined by the forms of the definitions of these quantities, we may have a suspicion as to how far these relations express what is most fundamental in the whole subject. Such a suspicion is entirely justified, and to illustrate the extent to which it is justified, we shall show that it is possible to make relations of the type of the circuital relations play a role in the description of a highly arbitrary law.

Let us cease to think of all electrical phenomena, and divide all space into a net-work of infinitesimal boxes which may, in the initial instance, be rectangular if we wish. Let us now write a set of perfectly arbitrary numbers on cards, and throw one into each box. Let the boxes now move about in any manner we like to assign. Let them twist into all sorts of shapes, some becoming very much elongated and others flattened, subject only to the condition that the network of boxes always remains unbroken, and that the boxes were originally chosen so small that all their dimensions remain infinitesimal. It will also be convenient to have the dimensions of these boxes of an order of magnitude lower than that of the differential elements of volume which are concerned in the specification of the ultimate differential equations of the medium. It will be obvious that while there may be some law according to which the sum of the numbers assigned to any small volume of space may be expressible in terms of the distributions of the numbers throughout space and of the motions of the boxes, the law will be entirely of our own creation. Nevertheless, if we let W represent the volume of some box which contains a number N, and if we define a quantity ρ by the equation

$$\rho=\frac{\Sigma N}{\Sigma W}$$

where ΣW is taken for an element of volume $d\tau$, and ΣN represents the corresponding summation of the numbers therein, and if we define two quantities φ and U by the equations

$$4\pi cU=\int\int\int\frac{[\rho v]}{r}\,d\tau,\quad 4\pi\varphi=\int\int\int\frac{[\rho]}{r}\,d\tau$$

where v is the velocity of a box, and c is a constant; and, if we then define two quantities E and H by the equations

$$E=-\frac{1}{c}\frac{\partial U}{\partial t}-Grad\ \varphi,\quad H=Curl\ U$$

the quantities E and H so defined will satisfy the circuital relations, for the necessary relation (33) which must exist between U and φ for this to be true, and which is the analytical equivalent of the equation of continuity, is automatically taken care of by our having specified that the numbers in the boxes are to remain unchanged as the boxes move about. Having done all this it *might* of course be possible to express the law of the motion of the numbers at any point in terms of the fields E and H, in a manner analogous to that adopted in the force equation of electromagnetic theory, although the equation would naturally be quite different from that equation in form. It is rather disturbing to contemplate the fact that any other being who happened to discover this purely arbitrary law of our own creation might discover these circuital relations as playing an important part in its description, and might be led to regard the fact as very fundamental. It would seem that we have here an illustration of the fact that what we call the laws of nature are in very large measure the statement of the way in which we have decided to think about nature. Indeed, the whole claim of E and H to fundamentality in electro-magnetic theory lies in the fact that it is possible to describe the motion of an element of charge entirely in terms of itself and of E and H, in a fairly simple manner, without any specification of the motions of the remaining electrons, other than is involved in the specifications of E and H themselves.

Without pursuing in too great detail this rather futuristic way of thinking, we are driven to admit that the equations giving the fields in terms of the motions of the charges contain practically nothing which is not definition, and that the real law of electro-magnetic theory is not stated until the force equation is stated. So far we have spoken of this equation without any references to the hypothesis of electrons, except in so far as we have spoken of electrons for purposes of description.

The force equation in relation to electrons.—As already implied, the force equation is usually stated in the sense implied in the statement that the force exerted on the electron due to its motion is equal and opposite to the force due to the external field. Directly some hypothesis has been made as to the variation of shape, density, etc. with velocity, as for example the hypothesis which has been made in the electron suggested by Prof. Lorentz, it is ideally possible to calculate the force which the electron exerts upon itself. It turns out to be zero for all cases of uniform velocity, but it contains terms involving the acceleration of some specified point, (the center for example in the case of an electron of symmetry) and all the higher time derivatives of the motion. Since there is no reason to expect that, in the general case, the resultant acceleration of the electron will be in the direction of what we have artificially called the force which it exerts upon itself, our law of motion will, in general, when expressed vectorially in terms of the accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ in directions parallel to and perpendicular to the velocity assume the form

$$\text{External force} = m_1\,\mathbf{a}_1 + m_2\,\mathbf{a}_2 + \text{terms involving higher time derivatives etc.} \tag{45}$$

the quantities m_1 and m_2 which are in general functions of the velocities are called the longitudinal and transverse masses, so that by this analytical artifice, what was originally a very unusual way of expressing an equation of motion of an electron has been forced into something like the familiar form

$$\text{External force} = \text{mass} \times \text{acceleration}. \tag{46}$$

The experimental verification of any hypothesis with regard to the electron rests upon the determination of the force which it exerts upon itself due to its motion, and is usually restricted to the determination of the masses in equation (45), and usually the transverse mass at that. It would appear that here our experimental knowledge of the situation is most incomplete. The postulation of the restricted theory of relativity, on the bases of experiments supporting it, carries with it the conclusion that the longitudinal and transverse masses m_1 and m_2 are of the form

$$m_1 = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \tag{47}$$

$$m_2 = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \tag{48}$$

the second of which expressions has been experimentally verified by the experiments of Bucherer. These are of course the masses of the Lorentzian electron; and, for the case of axes which coincide with neither the tangent nor the normal to the path, the equation of motion is

$$E_2 + \frac{[uH_2]}{c} = m_0 \frac{d}{dt}\left(\frac{u}{\left\{1-\frac{(u_x^2+u_y^2+u_z^2)}{c^2}\right\}^{\frac{1}{2}}}\right) \tag{49}$$

to the extent that the higher derivatives referred to in equation (45) are negligible. Equation (49) is of course consistent with the values given by (47) and (48) for the longitudinal and transverse masses. It is to be observed that while the postulation of the restricted theory of relativity provides us with some information as to the equation of motion of the electron, it does not give this equation completely. If we should assign some form for the force which the electron exerts upon itself in terms of its acceleration, rate of change of acceleration, and all higher derivatives, when the velocity is zero, or if we should do the equivalent of this by assigning the nature of the electron and the mutual relations of the higher time derivatives of the motion of its various parts, supposing this relation to be unique, then, the theory of relativity would have the power to tell us what the said force exerted by the electron upon itself would be when the velocity was not zero. Apart from this, however, it would be powerless to suggest anything as regards the part played by the higher time derivatives. (See Appendix, Note 5.)

The equation of motion of the electron remains the most important thing requiring experimental investigation, and the part most in need of it. It is the key to the whole situation, and is yet something of which we know least. Of course for slow motions, and for fields uniform over the electron, there is no difficulty, but, for high speeds, or more particularly rapid changes of speed, and sharply varying fields it would seem that we lack much. In particular, attention may be called to one case.

It is possible to obtain in the laboratory electronic speeds so high that the field of the electron tends to become concentrated in an equatorial plane. Suppose the velocity is so great in some particular case that the variation of the field is appreciable over distances from the equatorial plane comparable with the diameter of an electron. One's whole physical instinct revolts against saying that the electron upon which this field acts will, for any given state of motion of its center, adjust its shape in exactly the same way as if the external field were uniform over it. The process of determining the motion by equating the integrated external force to the integrated internal force seems now extremely crude. Yet, we have absolutely no theoretical criterion to guide us in assigning relative motion to parts of the electron, unless something could be developed on the lines of a postulate that the resultant vector $(E+[vH]/c)$ must be zero at each point. To attempt to apply ordinary mechanical concepts to the inside of the electron is rather like trying to explain the tenacity of a brick by considering it as made up of a number of houses.

The Equations of Conservation of Momentum and Energy

Significance of the equations.—In view of the very fundamental role played by momentum and energy in the discussion of some of the most vital aspects of modern physics, a word concerning these quantities will not be out of place. In the first place it must be emphasized that the equations representing the principles of conservation of momentum and energy are not founded upon any experimental basis as to the fundamental necessity for the conservation of those quantities. Given the electromagnetic equations they follow as a result of pure algebra, and their derivation is really the equivalent of finding the quantities which must be called work, energy-density, and momentum, in order that the quantities so defined shall be related by the same equations as those which relate the correspondingly named quantities in ordinary dynamics. The only justification for naming the quantities $[E.H]/c$, $(E^2+H^2)/2$, $\rho(v.\overline{E+[vH]/c})$, density of momentum, density of energy, rate of doing work per unit volume, is to be found in the fact that the electromagnetic equations give to these quantities the properties which the correspondingly named quantities in ordinary dynamics are supposed to possess. It may happen, and actually does happen in the case of work and energy, that the quantities to which these names are assigned are chosen as the result of the consideration of a rather specialized type of problem. Having once named them one is inclined to seize upon them as quantities of fundamental importance. The glorious and honorable titles work, energy and momentum which have been assigned to them tend to cause us to forget their birth, and to endow them, on the basis of these titles, with properties which they never obtained from their parents. Then, when we consider some problem which is of a more general type than that which was in mind when the names were given, and find that these quantities do not have the properties which we expected of them, we are apt to be greatly astonished and claim that our theories are doing something outrageous in calling for a revision of the principle of the conservation of energy. The truth of the matter of course lies in the fact that if in any special case we should find any discrepancy of the kind stated, it simply means that, to the degree of generality implied by the inclusion of that special case, the quantities in question had no right to have the names work, energy, etc., assigned to them. The only proper basis for choosing the quantities which shall be given these names is that they shall be the quantities to which our scheme of laws gives, in all cases, the properties which we desire of them.

Localization of energy.—Difficulties are also apt to arise from too strong visualization of the supposed seat of the energy in the medium. Thus, to take the very simplest case, that of electrostatics, we recall that it is customary to say that the energy is distributed throughout the volume to the extent $E^2/2$ per unit volume. The origin of this conclusion may be summarized as follows: We first consider a system of charged bodies, and assuming only that the forces are derivable from a potential V we calculate for the energy the expression $W=\frac{1}{2}\Sigma QV$. Extending this to the case of continuous distributions with volume densities ρ and surface densities σ it becomes replaced by

$$W=\frac{1}{2}\int\int\int\rho V d\tau+\Sigma\frac{1}{2}\int\int\sigma V dS \qquad (50)$$

The next thing is to replace ρ by Div E, which, combined with the assumption that E has no curl, is the equivalent of assuming the law of inverse squares. On applying Green's theorem to the volume integral thus formed, a new volume integral $\frac{1}{2}\int\int\int E^2 d\tau$ is formed, and a surface integral is thrown out at the boundary of each of the regions which are bounded by the surfaces. With the exception of the bounding surface at infinity, each surface gives rise to two surface integrals, one on account of that surface functioning as the outer boundary of the region which it incloses, and the other on account of its functioning as the inner boundary of the region which lies outside it. These two surface integrals when combined so as to give one surface integral for each surface turn out to just cancel individually the several surface integrals of equation (50). They do this on account of the fact that σ is just equal to the discontinuity in E_n at a point on the surface. The result of all this is that the volume integral $\frac{1}{2}\int\int\int E^2 d\tau$ remains as the sole representative of the energy. The form of this expression strongly suggests the assignement of energy $E^2/2$ to each unit of volume, and so strong is this suggestion that undoubtedly a good many physicists regard the demonstration as *proving* that the energy is to be so located, or at least regard any objection to the acceptance of the demonstration as a mere splitting of hairs. The expression representing the energy entirely as a volume integral throughout space is not confined to the case where there is a finite volume density, but holds even in the case where the charges are entirely confined to the surfaces. In this case, of course, the volume integral of (50) is zero, but it can nevertheless be split by Green's theorem into two parts, a volume integral $\frac{1}{2}\int\int\int E^2 d\tau$, and a set of surface integrals which cancel the surface integrals of (50) leaving the expression $W = \frac{1}{2}\int\int\int E^2 d\tau$ as before.

In order to demonstrate the artificiality of the process by which the above conclusions as to the location of the energy are reached, it may not be out of place to consider an actual possibility as regards another method of distributing the energy.

Imagine a mechanism by which two bodies are connected by a sort of screw, and that, as each body moves along the screw towards the other, it winds up, by means of a nut engaging with the screw, a spring which is located within itself. A means is supposed to be provided for preventing the body itself from rotating. One body will of course experience a force as it is pushed towards the other, and it would not be impossible to arrange the pitch of the screw and the nature of the springs so that this force was that of the inverse square. We can further imagine a mechanism of this kind extended to each pair of bodies in our whole system. The actual mechanism would be, of course, extremely complicated in practice, but it would not appear that its realization would present any *logical* difficulties. Now if all this mechanism of screws and springs be rendered invisible, leaving only the bodies visible, and if the whole system be presented to somebody else who knows nothing of the mechanism, he will proceed to study it in the same sort of way that we study electrical phenomena. He will find that the bodies exert upon each other forces which are derivable from a potential. This is all that he will need to construct the expression $W=\Sigma\frac{1}{2}QV$ for the energy. He will then extend this to $\frac{1}{2}\int\int\int\rho V d\tau$ to include the case of continuous distributions. He will discover that the forces are governed by the inverse square law, which is all that he needs to enable him to write ρ as proportional to Div E. He will then be able to carry out the transformation by Green's theorem arriving at the expression $W=\frac{1}{2}\int\int\int E^2 d\tau$, and will have proved possibly to his own entire satisfaction that the potential energy is located in the medium to the extent $E^2/2$ per unit volume whereas we, who have constructed the mechanism, know that in the sense that we understand the term location of energy, the energy is really in the springs which are hidden in the bodies and is not to be found in the medium outside of the bodies.

With these preliminary general considerations we pass on to consider a few specific points in relation to the equations of energy and momentum.

The momentum equation.—The momentum equation is expressed in the form

$$\int\int\int\rho\left(E+\frac{[vH]}{c}\right)d\tau = -\frac{\partial}{\partial t}\int\int\int\frac{[EH]}{c}\,d\tau + \text{Surface integrals}$$

the volume integrals being taken throughout any specific volume over whose surface the surface integrals are taken. $[E\,H]/c$ is the momentum per unit volume. The equation as it stands, however, is of very little use. Its strength lies in the fact that it applies to any field satisfying the electromagnetic equations. If then we incorporate a hypothesis such as that embodied in the statement that the electron moves so that the force exerted on it by the external field is equal and opposite to the force which it exerts on itself, and if we remember that both E_1, H_1, the field of the electron, and E_2, H_2, the external field are separately solutions of the electromagnetic equations, we shall be able to write

$$\int\int\int\left(E_2+\frac{[vH_2]}{c}\right)\rho\,d\tau = -\int\int\int\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau \qquad (51)$$

and,

$$\int\int\int\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau = \frac{\partial}{\partial t}\int\int\int\frac{[E_1H_1]}{c}d\tau + \text{Surface integral involving } E_1 \text{ and } H_1 \qquad (52)$$

In this form the equation is of use, since, when the type of the electron and its motion are given, the right hand side can be calculated as a function of the motion, and the equation therefore relates the motion of the electron to the external field. In this form the equation does of course involve something drawn from experiment, or at any rate from additional hypothesis, viz: the statement of equality of external and internal force. It is indeed the analytical equivalent of that law. One useful service performed by the equation in this form is that of enabling us to calculate the electromagnetic masses, which are defined as the coefficients of the accelerations in the directions parallel to and perpendicular to the path, in the expression for the external force acting upon the electron. The particular use of the momentum equation for this purpose over that involving a direct calculation of the force exerted by the electron on itself lies in the fact that while, in both cases, we strictly arrive at a result in which the external force is not completely expressible in terms of the velocity alone, all higher time derivatives being in fact present, the momentum equation by making use in the most direct manner of the assumption of the quasi stationary principle is able to arrive more directly at a result which is correct as far as the acceleration terms are concerned.

The quasi stationary principle amounts to stating that, in calculating the momentum

$$M=\int\int\int\frac{[EH]}{c}\,d\tau$$

we may, to the approximation cited, calculate it at each instant on the assumption that the field of the electron is the same as it would have been if the electron had been moving for an infinite time with the velocity which it has at the instant. On this assumption M is a function of v only

$$M=vf(v)$$

Hence, for the component parallel to the x axis

$$M_x=v_xf(v)$$

As regards the integral through all space we may replace partial $\partial/\partial t$ by d/dt , so that

$$\frac{\partial M_x}{\partial t}=\frac{dM_x}{dt}=\dot{v}_xf(v)+v_x\,\dot{v}f'(v)$$

If we choose the axis of x perpendicular to v, so that $v_x=0$, we have

$$\frac{dM_x}{dt}=\dot{v}_xf(v)=\left(\frac{M}{v}\right)\dot{v}_x$$

Hence the transverse mass m_2 is given by

$$m_2=\frac{M}{v}$$

If we choose the axis of x parallel to v, so that $v_x=v$

$$\frac{dM_x}{dt}=\dot{v}f(v)+v\,\dot{v}\,f'(v)$$

$$=\dot{v}\frac{d}{dv}\Big\{vf(v)\Big\}=\dot{v}\frac{dM}{dv}$$

showing the longitudinal mass m_1 to be given by

$$m_1=\frac{dM}{dv}$$

It is to be particularly remarked that it is only because the equations of motion of the electron have been forced into the form force = mass × acceleration that they have assumed this form at all; and, as already pointed out, we have no right whatever to expect any direct or simple relation between these particles which constitute the electrons and those hypothetical particles of constant mass to which, as Poincaré has shown us, any dynamical system may be referred.

The energy equation.—Remarks exactly analogous to the above may be made in relation to the energy equation[1]

$$\iiint\left(\overline{v.\,E+\frac{[vH]}{c}}\right)\rho d\tau=-\frac{\partial}{\partial t}\iiint\left(\frac{E^2+H^2}{2}\right)d\tau+c\iint[EH]_n dS \quad (53)$$

Here again, the equation applies to the field of the electron alone as well as to the whole field, so that if subscript unity refers to the field of the electron we have

$$\iiint\left(\overline{v.E_1+\frac{[vH_1]}{c}}\right)\rho d\tau=-\frac{\partial}{\partial t}\iiint\left(\frac{E_1^2+H_1^2}{2}\right)d\tau$$
$$+c\iint[E_1H_1]_n dS \quad (54)$$

If, *and only if*, all parts of the electron move with the same velocity we are at liberty to take the v outside of the integral on the left hand side, and then, by utilizing the fundamental assumption as to the equality of the external force on the electron and the force which it exerts on itself, we arrive at the relation

$$v\iiint\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau=\frac{\partial}{\partial t}\iiint\left(\frac{E_1^2+H_1^2}{2}\right)d\tau-c\iint[E_1H_1]_n dS \quad (55)$$

which expresses a relation between what we may call the work done by the external forces on the electron, the increase of energy of the field

[1] As pointed out in Note 2, of the appendix, and also earlier in this report, the expression $\frac{1}{2}\int\int\int(E^2+H^2)\,d\tau$ is not the only one which is used in electrodynamics as the appropriate expression for the energy.

of the electron, and the flow of energy of the field of the electron through the boundary.

It is to be observed, however, that the whole justification for calling the left hand side the work done per second on the electron by the external forces, and for the expectation of any relation of the above kind between the work as so defined and the quantity which has been called the energy of the electron, rests upon the possibility of taking v outside of the integral of equation (54). It is only for such a case, i. e., one where the velocity of all parts of the electron is the same, that the names work and energy are appropriately given to the quantities to which they have been applied.

The Lorentzian electron and the conservation of energy.—Now it is usually argued that in the case of the Lorentzian electron, a difficulty exists as to the application of the principle of conservation of energy; and the difficulty is this:

If we calculate the longitudinal electromagnetic mass of the electron we find

$$m_1=\frac{e^2}{6\pi ac^2}\left(1-\frac{v^2_0}{c^2}\right)^{-3/2} \tag{56}$$

where a is the radius of the spherical shell of charge which constitutes the electron when at rest, v_0 the velocity of the center of symmetry of the electron, e the electronic charge, and c the velocity of light. The external force parallel to the velocity is obtained by multiplying the above value of m_1 by dv_0/dt, and the quantity which is called the rate of doing work by, or activity of, the external force and is obtained by multiplying the result by v_0 is

Activity of external force =

$$v_0\int\int\int\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau=\frac{e^2}{6\pi ac^2}\left(1-\frac{v_0^2}{c^2}\right)^{-3/2}v_0\frac{dv_0}{dt} \tag{57}$$

On the other hand the rate of increase of the energy of the electron as calculated from the first member of the right hand side of (54), with the boundary extended to infinity is

Rate of increase of energy of electron =

$$\frac{e^2}{6\pi ac^2}\left(1-\frac{v_0^2}{c^2}\right)^{-3/2}v_0\frac{dv_0}{dt}+\frac{e^2}{24\pi ac^2}\left(1-\frac{v_0^2}{c^2}\right)^{-1/2}v_0\frac{dv_0}{dt} \tag{58}$$

Expression (57) is less than (58)[1] by

$$\frac{e^2}{24\pi ac^2}\left(1-\frac{v^2_0}{c^2}\right)^{-1/2}v_0\frac{dv_0}{dt},$$

and the inference is that the Lorentzian electron violates the principle

[1] The discrepancy has nothing to do with any possible value of the third integral in (54) when taken over the infinite boundary or with the application of the so called quasi stationary principle.

of conservation of energy. With the object of removing this difficulty, Lorentz has gone so far as to invoke the energy associated with certain non-electrical forces first introduced by Poincaré,[1] for the purpose of accounting for the equilibrium of the electron in its spherical and flattened forms.

Now when we recall that the principles upon which (57) is obtained involve nothing but the momentum equation in the form (52), combined with the *definition* that the activity of the external force is to be taken as the external force multiplied by the velocity of the center of the electron, and when we also recall the origin of the expression for the energy, we see that an inconsistency is algebraically impossible provided that all the quantities are correctly named. The fact is that, in the case of the Lorentzian electron, we have a situation where all parts of the electron do not move with the same velocity. The electron contracts while in motion, so that, in the case of acceleration in the direction of the velocity, for example, its fore end moves at each instant slower than its center and its back end moves faster. The situation is one where we are not justified in taking v outside of the integral in equation (54) and we are consequently unable to make use of the equality of the external force and the force which the electron exerts upon itself to obtain an equation such as (55). As a matter of fact the product of external force and velocity of the center of the electron is a quantity which our equations do not endow with any very important property in the case of the Lorentzian electron, however much its form may tempt us to expect important properties in it; and it is important to realize that there is no other source from which this quantity may expect to derive important properties.

It will perhaps be of value to probe this matter in detail in order to ascertain just what the equations do have to say regarding it. Using as before E_1 and H_1 to denote the field of the electron, equation (54) follows as an algebraic necessity from the electromagnetic equations. Let us put

$$v = v_0 + \Delta v$$

where v_0 is the velocity of the center of symmetry of the electron, and Δv is the algebraic excess of the velocity of a point on the surface over the value v_0. We then have,

$$\iiint\left(v.\ \overline{E_1+\frac{[vH_1]}{c}}\right)\rho d\tau = \iiint v_0\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau + \iiint \Delta v\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau$$

[1] Sur la Dynamique de l'electron, Rendiconti del Circolo Mathematico di Palermo 21, p. 129, 1906.

It is now possible to take v_0 outside of the integral in which it occurs, and we may then legitimately make use of the equality of the external force on the electron and the force which the electron exerts upon itself, and may write

$$\iiint\left(v.\,E_1+\frac{[vH_1]}{c}\right)\rho d\tau=-v_0\iiint\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau+\iiint\Delta v\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau$$

substituting in (54) we find

$$v_0\iiint\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau=\iiint\Delta v\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau+\frac{\partial}{\partial t}\iiint\left(\frac{E_1^2+H_1^2}{2}\right)d\tau \qquad (59)$$

where we have omitted the surface integral in (54) on the basis that it is taken over an infinitely distant boundary.

Thus we see that the so called activity of the external force as represented by the left hand side of (59) should *not* be equated to the rate of increase of energy as represented by the second term on the right hand side of (59). Equation (59) is not a denial of the conservation of energy, it is simply a criterion as to what should be called work and energy in order that these quantities shall be related in the manner which their names signify, and the equation shows us that the member on the left hand side is not the quantity to call the rate of doing work on the electron by the external force if the second member on the right hand side is to be called the total rate of increase of energy of the electron.

If we calculate the magnitude of the first member on the right hand side of (59), we find for it (see Appendix, Note 6) the value

$$-\frac{e^2}{24\pi ac^2}\left(1-\frac{v_0^2}{c^2}\right)^{-1/2}v_0\frac{dv_0}{dt}$$

so that, in view of the value which has been quoted in equation (58) for the second member on the right hand side; we have

$$v_0\iiint\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau = -\frac{e^2}{24\pi ac^2}\left(1-\frac{v_0^2}{c^2}\right)^{-1/2}v_0\frac{dv_0}{dt}+\frac{e^2}{6\pi ac^2}\left(1-\frac{v_0^2}{c^2}\right)^{-3/2}v_0\frac{dv_0}{dt}+\frac{e^2}{24\pi ac^2}\left(1-\frac{v_0^2}{c^2}\right)^{-1/2}v_0\frac{dv_0}{dt}$$

Thus the term which has arisen from the member of (59) which involves Δv just cancels that portion of (58) which has been the subject of all the discussion, leaving an equation which is perfectly consistent with equation (57) which was built up from the momentum equation. Of course it is algebraically impossible that the result could have been otherwise.

Material Media

The Lorentzian equations.—The impossibility of dealing with each individual molecule in a medium naturally makes it desirable to formulate equations which express the laws as far as possible in terms of quantities which have a significance only in the macroscopic sense. With this end in view, Lorentz[1], starting with the field equations for free aether has proceeded to build up average values for the quantities E, H, ρv, etc., occurring in the equations, and has arrived at the following system of equations applicable to a moving material medium:

$$\frac{1}{c}\left(\frac{\partial E}{\partial t}+\rho u+j+\frac{\partial P}{\partial t}+Curl\,[Pu]+c\,Curl\,m\right)=Curl\,B \qquad (60)$$

$$-\frac{1}{c}\frac{\partial B}{\partial t}=Curl\,E \qquad (61)$$

$$Div\,B=0 \qquad (62)$$

where the quantities have the following significance:

j is the conduction current density.

u represents the velocity of the material medium at any point.

ρ is the net density macroscopically considered in a macroscopic volume element.

E and B are the average values of the true electric and magnetic fields, the averages being taken throughout the macroscopic element of volume.

P is the polarization. In the Lorentzian sense, it is defined in a way which is rather different from what one would expect on the basis of an analogy with Poisson's theory of dielectric media at rest.

P is in fact defined in the following way: Let us divide the electricity in a macroscopic element of volume into its positive and negative parts which we shall keep separate in our mind's eye, making no attempt to associate any one positive element with some neighboring negative element for the purpose of forming a doublet. Multiply the macroscopic density ρ_0 of the positive electricity by the amount r by which it has been displaced from some standard position, and call $p_+=\rho_0 r$ the polarization corresponding to the positive electricity. Build up a similar quantity for the negative electricity. Then, vectorially

$$P=p_+ + p_- \qquad (63)$$

The characteristic feature of this polarization which distinguishes it from that defined in terms of doublets, is that it speaks of the density

[1] Enzyk. der Math. Wiss. v. 2, pp. 200–9.

of the positive and negative electricity at *one and the same point* (macroscopically considered), and concerns itself with the amounts by which they *have been* displaced *to* that point, whereas the point to which a doublet is referred is the point *from which* the charge has been displaced. This characteristic difference has a very important bearing upon the manner in which it is possible to average the ρv terms.

The quantity m, the magnetization, is defined as

$$m = m_1 + m_2 \tag{64}$$

where

$$cm_1 = \tfrac{1}{2}\,\rho_0\,[r_1\,\dot{r}_1] \text{ and } cm_2 = \tfrac{1}{2}\,\rho_0\,[r_2\,\dot{r}_2] \tag{65}$$

If we define an additional quantity D by the relation

$$D = E + P \tag{66}$$

this quantity satisfies the relation

$$Div\ D = \rho \tag{67}$$

It is also customary to define a quantity h, by the relation

$$h = B - m \tag{68}$$

Now an examination of the definitions of the above quantities will show that, with the exception of perhaps u, not one of the quantities is capable of experimental realization in terms of its definition. In other words, we could not utilize the definitions for the measurement of the quantities. In view of this, it is a matter of importance to inquire just what the equations mean. We can get some light upon this matter by going through a process of solving them analogous to that adopted in the equations for free aether. If we do this, we can show without difficulty that we arrive at the relations

$$E = -\frac{1}{c}\frac{\partial U}{\partial t} - Grad\ \varphi \tag{69}$$

$$B = Curl\ U \tag{70}$$

where U and φ are of the same form as (32) except that in the expression for U, ρv is replaced by

$$J = j + \rho u + \frac{\partial P}{\partial t} + Curl[Pu] + c\ Curl\ m \tag{71}$$

and in the expression for φ, ρ is replaced by $\rho - Div\ P$. The last named result is only what might have been expected in view of the fact that the $-Div\ P$ repesents the fictitious volume charge density.

These expressions by themselves would not be complete however. For in the process of building up the equations, it becomes implicitly involved that, on account of the displacements of the positive and negative electricity within the volume elements, there is left over at the bounding surface of the medium a charge which is everywhere equal to P_n, the outward normal component of P at the surface. In the case of a transition from one medium to another it is the discontinuity in P_n which is involved. Then, in a case of this kind, it is not in the spirit of a macroscopic analysis to regard a surface density of real charge as a limiting case of a volume density, but is preferable to leave it in the form of a surface density. On these accounts there will be two additions to the scalar potential, so that, using $\overline{P}_n$ to denote the discontinuity of P_n,

$$4\pi\varphi=\iiint\frac{[\rho-Div\ P]}{r}\,d\tau+\iint\frac{[\sigma+\overline{P}_n]}{r}\,dS \tag{72}$$

Again, both the real and fictitious surface charge will contribute the equivalent of a surface current sheet, and both of these will consequently contribute to the vector potential. Furthermore, corresponding to the volume distribution of magnetization, there will be a surface distribution, which will also contribute to the vector potential. When all of these contributions are taken into account, the complete expression for the vector potential is found to be

$$4\pi c\,U=\iiint\frac{\left[j+\rho u+\frac{\partial P}{\partial t}+Curl\ [Pu]+c\ Curl\ m\right]}{r}\,d\tau \tag{73}$$
$$+\iint\frac{[u\sigma+u\overline{P}_n+c\ Curl\ [n\overline{M}]]}{r}\,dS$$

where the square brackets around the complete numerators of the integrals for φ and U indicate retarded potentials, but the other square brackets of course indicate vector products, and where $\overline{P}_n$ and $\overline{M}$ indicate the discontinuities of P_n and m at the boundaries. The n in the expression $Curl\ [n\overline{M}]$ refers to the unit vector in the direction of the normal.

Although, as we have already remarked, the origin and definition of the quantities P, E, B, etc., are not such that we can experimentally realize the quantities in terms of those definitions, nevertheless, having formed the above equations we may forget all about the origin of the expressions, and formulate the laws as follows:

Divide all space into elements of volume, and all surfaces separating different media into elements of surface. Then, it is possible to assign to each element of volume four numbers ρ, j, P, and m, the first two satisfying the relation

$$-\frac{\partial\rho}{\partial t}=Div\ (j+\rho u) \tag{74}$$

and to each element of surface, a number σ such that if φ and U be defined by equations (72) and (73) and E and B by the equations (69) and (70), the values of E and B so obtained for a point in free aether may be used in the force equation for the purpose of calculating the motion of any electron or charged body which can be handled by that equation.

The essential point of this formulation lies in the fact that no attempt is made to define the quantities P, E until after the truth of the law has been recognized, there is no measurement of forces within cavities or the like. Moreover, as regards the expression of what experiment can actually reveal, it would be possible to eliminate E and B from the discussion entirely in the manner indicated on page 28, for the case of the equations for free aether. It is an advantage to retain these quantities, however, not merely for analytical convenience, but also because it is in terms of them that the special properties of any particular medium are expressed. Thus in the above formulation, no statement is made as to any relation between such quantities as P and E other than those implied in the general equation. In practice, however, we know that P does depend upon E and possibly upon the other quantities B, u, etc. In the simplest case, of course $P=(K-1)E$. These constitutive relations, concerning which the general law tells us nothing are very important from the practical standpoint, since, if known, they limit the scope of generality in any particular problem, and serve to give enough relations to make the solution definite without the specification of more than a limited number of conditons. For the constitutive relations we must rely upon experiment in the case of any particular substance, although the restricted theory of relativity lends a valuable guide in requiring that, in the expression of these relations, the quantities related shall have the mathematical characteristics of being 4-vectors (see Appendix, Note 7.)

It may be of interest to conclude this part of the subject by giving a derivation of the Lorentzian equations for a material medium, by a process slightly different from that generally adopted, and one moreover following more closely the usual definition of polarization as commonly met with in the statical case.

A derivation of the Lorentzian equations for moving media.—As regards the determination of the field at a point outside of the medium, a point in a cavity for example, everything is known provided that we know the complete expressions for the scalar and vector potentials. It will be convenient to work with the non-retarded, or Maxwellian vector and scalar potential which we shall denote by A and ψ. It is then our purpose to determine the forms of A and ψ expressed in terms of *macroscopic* distributions, which shall have the same values as the A and ψ given by the sub-molecular analysis. Using subscript zero to denote quantities considered in the sub-molecular sense, and omitting the subscript when the quantities are to be understood in the macroscopic sense, we have

$$4\pi cA=\iiint\frac{(\dot{E}_0+\rho_0v_0)}{r_0}d\tau_0 \quad , \quad 4\pi\psi=\iiint\frac{\rho_0}{r_0}d\tau_0$$

The integrals are extended throughout all space, although, as regards contributions from points in free aether, there is no need to distinguish between a fine-grained and a macroscopic analysis.

Let ν be a macroscopic element of volume, and let Σ refer to summation taken throughout ν. Then

$$A=\frac{1}{4\pi c}\iiint\left\{\frac{1}{\nu}\Sigma\frac{(\dot{E}_0+\rho_0v_0)}{r_0}d\tau_0\right\}d\tau \tag{75}$$

where the integral is now taken with regard to the macroscopic elements $d\tau$, the subscript being omitted to denote the macroscopic nature of $d\tau$. The distinction between $d\tau$ and ν is merely formal, but $d\tau_0$ is still retained for the purpose of the summation denoted by Σ.[1]

If E only varied throughout the macroscopic volume element by quantities of the first order of macroscopic small quantities, we could write

$$\Sigma\frac{\dot{E}_0}{r_0}d\tau_0=\frac{1}{r}\Sigma\dot{E}_0d\tau_0$$

Since, in view of the sharp changes in charge density, this may not be the case, however, we must write

$$E_0=E+\Delta E$$

where E is the true average value of E_0 throughout the volume element, and ΔE may be finite although $\Sigma\Delta E=O$. We thus have

$$\Sigma\frac{\dot{E}_0}{r_0}d\tau_0=\frac{\nu\dot{E}}{r}+\Sigma\frac{\Delta\dot{E}}{r_0}d\tau_0$$

Thus, if u_x, u_y, u_z is the velocity of the matter, and if we separate off the parts j and ρu which are to be considered as conduction and convection current, we have

$$4\pi cA=\iiint\frac{(j+\rho u+\dot{E})}{r}d\tau+\iiint\left\{\frac{1}{\nu}\Sigma\frac{(\rho_0v_0+\Delta\dot{E})}{r_0}d\tau_0\right\}d\tau \tag{76}$$

where ρ_0v_0 refers now only to the so called bound electricity.

Let us divide the electricity in the elements up into equal positive and negative parts, and let us associate these parts in pairs. The pairs may be taken in any way we please; but, for the purposes of a mental picture it will be convenient to think of corresponding elements of a

[1] The only reason for using Σ in place of integral signs is to avoid confusion between the summation throughout the element ν, and the summation or integration, in which ν itself figures as a volume element.

doublet. Corresponding to each pair, we shall associate a point in ν which has an unvarying relation to the pair. Thus, it may be the point which moves so that it is always half way between the elements of a pair. It shall be the point to which we shall refer the position of the doublet, considered as a single entity.

Let x, y, z, be the displacement of this point from the position which it occupied in the medium at some definite instant.

Let $\frac{1}{2}(\xi, \eta, \zeta)$ be the displacement of the positive element from it.

Let $-\frac{1}{2}(\xi, \eta, \zeta)$, be the displacement of the negative element from it, and let the corresponding dotted quantities refer to the corresponding velocities relative to the respective origins from which the displacements are measured. Then

$$4\pi cA_x = \iiint \frac{(j_x+\rho u_x+\dot{E}_x)}{r}d\tau + \iiint\left[\frac{1}{\nu}\Sigma\left\{\frac{\epsilon(u_x+\dot{x}+\frac{1}{2}\dot{\xi})}{r_0} - \frac{\epsilon(u_x+\dot{x}-\frac{1}{2}\dot{\xi})}{r_0} + \frac{\Delta\dot{E}}{r_0}d\tau_0\right\}\right]d\tau \tag{77}$$

where ϵ here denotes the positive value of the element of charge, and u_x refers to the representative point for the pair, but r_0 refers, in each case, to the actual position of the charge. Thus,

$$4\pi cA_x = \iiint \frac{(j_x+\rho u_x+\dot{E}_x)}{r}d\tau + \iiint\frac{1}{\nu}\left[\Sigma u_x\epsilon\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s + \Sigma\frac{\dot{\xi}\epsilon}{r_0} + \Sigma\dot{x}\epsilon\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s + \Sigma\frac{\Delta\dot{E}}{r_0}d\tau_0\right]d\tau \tag{78}$$

where $\delta s^2 = \xi^2+\eta^2+\zeta^2$

If p_x, p_y, p_z, refer to the component moments of a single pair, i.e. if

$$p = \epsilon\times(\text{separation of the doublet elements})$$

$$u_x\epsilon\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s = u_x\epsilon\left\{\frac{\partial}{\partial x}\left(\frac{1}{r_0}\right)\xi + \frac{\partial}{\partial y}\left(\frac{1}{r_0}\right)\eta + \frac{\partial}{\partial z}\left(\frac{1}{r_0}\right)\zeta\right\} = p_x u_x\frac{\partial}{\partial x}\left(\frac{1}{r_0}\right) + p_y u_x\frac{\partial}{\partial y}\left(\frac{1}{r_0}\right) + p_z u_x\frac{\partial}{\partial z}\left(\frac{1}{r_0}\right) \tag{79}$$

Since u_x and r_0 are constant to the first order of macroscopic small quantities throughout the volume ν, if we write

$$\nu(P_x,\ P_y,\ P_z) = (\Sigma p_x d\tau_0,\ \Sigma p_y d\tau_0,\ \Sigma p_z d\tau_0)$$

we have[1]

$$\frac{1}{\nu}\Sigma u_x\epsilon\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s = P_x u_x\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + P_y u_x\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + P_z u_x\frac{\partial}{\partial z}\left(\frac{1}{r}\right) \tag{80}$$

[1] We may here parenthetically remark that, by considering the case u = constant, we see that the last two groups of three terms of (81), when integrated throughout the volume, represent respectively the contributions of the fictitious volume and surface charges to the vector potential. We might expect these to show themselves somewhere.

$$= -\frac{1}{r}\left\{\frac{\partial(P_x u_x)}{\partial x}+\frac{\partial(P_y u_x)}{\partial y}+\frac{\partial(P_z u_x)}{\partial z}\right\} \\ +\frac{\partial}{\partial x}\left(\frac{P_x u_x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{P_y u_x}{r}\right)+\frac{\partial}{\partial z}\left(\frac{P_z u_x}{r}\right) \tag{81}$$

Considering now the third member on the right hand side of (78), we have

$$\frac{1}{\nu}\Sigma\frac{\dot{\xi}\epsilon}{r_0}=\frac{1}{\nu r}\Sigma\dot{\xi}\epsilon=\frac{1}{r}\left[\frac{\partial P_x}{\partial t}+\frac{\partial}{\partial x}(P_x u_x)+\frac{\partial}{\partial y}(P_x u_y)+\frac{\partial}{\partial z}P_x u_z)\right] \tag{82}$$

The last three terms are added because the partial $\partial P_x/\partial t$ is equal to $\frac{1}{\nu}\Sigma\dot{\xi}\epsilon$ (i.e. the rate of increase of moment per cc. in the *fixed* volume element, on account of the rate of separation of the doublet elements within it) plus the rate at which moment enters the fixed volume element on account of the motion of the medium.

Adding the results obtained in (81) and (82), and substituting in (78), we have

$$4\pi cA_x=\iiint\frac{j_x+\rho u_x+\dot{E}_x)}{r}d\tau+\iiint\frac{1}{r}\left\{\frac{\partial P_x}{\partial t}+\frac{\partial}{\partial y}(P_x u_y-P_y u_x)\right. \\ \left.-\frac{\partial}{\partial z}(P_z u_x-P_x u_z)\right\}d\tau+\iiint\left\{\frac{\partial}{\partial x}\left(\frac{P_x u_x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{P_y u_x}{r}\right)\right. \\ \left.+\frac{\partial}{\partial z}\left(\frac{P_z u_x}{r}\right)\right\}d\tau+\iiint\frac{1}{\nu}\left\{\Sigma\dot{x}\epsilon\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s+\Sigma\frac{\Delta\dot{E}_x d\tau_0}{r_0}\right\}d\tau \tag{83}$$

$$4\pi cA_x=\iiint\frac{(j_x+\rho u_x+\dot{E}_x)}{r}d\tau+\iiint\frac{\left\{\frac{\partial P_x}{\partial t}+curl_x[Pu]\right\}d\tau}{r} \\ +\iiint\left\{\frac{1}{\nu}\Sigma\dot{x}\epsilon\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s\right\}d\tau+\frac{1}{\nu}\iiint\left\{\Sigma\frac{\Delta\dot{E}_x}{r_0}d\tau_0\right\}d\tau \\ +\iiint\left\{\frac{\partial}{\partial x}\left(\frac{P_x u_x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{P_y u_x}{r}\right)+\frac{\partial}{\partial z}\left(\frac{P_z u_x}{r}\right)\right\}d\tau \tag{84}$$

The last term, when integrated throughout the volume leads, by Green's lemma to

$$\iint\frac{u_x P_n}{r}dS$$

this integral being taken over the surface of the medium. We thus see that, as regards the vector potential A_x at an external point, the medium acts as though there were

(1) A current density $j_x+\rho u_x+\frac{\partial E_x}{\partial t}+\frac{\partial P_x}{\partial t}+\{curl\ [Pu]\}_x$ throughout the volume.

(2) A current corresponding to the motion with velocity u_x of electric density P_n on the surface.

(3) An effect represented by the remaining terms of (84) which we shall presently discuss.

We may here remark that, as regards (1) we are in agreement with Lorentz, except that our polarization P has a meaning slightly different from his. We have defined the moment per cc. in terms of the summation of the moments of doublets by which the displacements have been artificially represented. The moment of each of these doublets has been obtained by multiplying its pole-strength by the *separation* between the poles. As already remarked, Lorentz considers the positive electricity present in some volume element, and multiplies its amount by say the x component of the distance through which it *has been* displaced. He then does the same thing for the negative electricity, and after adding the two results together, calls the sum P_x. In other words, the corresponding positive and negative portions which figure in the definition of P_x are, as it were, superposed on each other. His analysis appears to avoid the term

$$\iiint\left\{\frac{\partial}{\partial x}\left(\frac{P_x u_x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{P_y u_x}{r}\right)+\frac{\partial}{\partial z}\left(\frac{P_z u_x}{r}\right)\right\}d\tau$$

occurring in (84), and he appears to obtain a current-density expression which is complete without it. This absence of the term is only apparent, however; for, in assigning different displacements to the positive and negative electricity in the volume elements, we are forced to have over, at the boundary of the medium, surface charges, if the displacements are to leave the medium uncharged as a whole. In a sense, we cannot average our current density throughout the macroscopic volume elements in such a way as to make the result the analytical equivalent of the fine-grained analysis, for, the effects which, in the fine-grained analysis, can be expressed in terms of the volume-element contributions alone can only be represented in the macroscopic analysis for the medium as a whole by contributions from the volume elements *together with* contributions from the surface. It may readily be shown that the surface charges which would be left over on the basis of the Lorentzian analysis would contribute by their motion exactly the term

$$\iint\frac{u_x P_n}{r}dS$$

which comes from the application of Green's lemma to the last term of (84). In the case of a boundary between two material media it is of course the discontinuity in P_n which is involved.

It remains to consider the term

$$\iiint\frac{1}{v}\left\{\Sigma\dot{x}_e\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s+\Sigma\frac{\Delta\dot{E}_s}{r_0}d\tau_0\right\}d\tau$$

Now it readily follows from the definition of P that $-Div\ P+\rho=Div\ E$, i. e., $\rho=Div(P+E)$.
Thus

$$Div(j+\rho u)+Div\frac{\partial(E+P)}{\partial t}=Div(j+\rho u)+\frac{\partial\rho}{\partial t}=0$$

on account of the equation of continuity. Hence the divergence of

$$j+\rho u+\frac{\partial(E+P)}{\partial t}+Curl\ [Pu]$$

is zero, and the divergence of the corresponding portion of A is zero. Since the divergence of A as a whole is zero it follows that

$$Div\int\int\int\frac{1}{\nu}\left[\left\{\Sigma\Delta\dot{r}e\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s\right\}+\Sigma\frac{\Delta\dot{E}}{r_0}d\tau_0\right]d\tau=0$$

where $\dot{x}$ in the expression under discussion has been replaced by $\Delta\dot{r}$ to symbolize the resultant direction of the vector. Now the term under discussion represents a contribution to A as a result of electronic motions which exist when u is zero. Moreover, it is not part of the bodily motion ρu, nor of the conduction current j, nor of the effective current arising from the rate of change of the polarization. It represents, in fact, the part of A arising from the magnetization of the medium. It is this residual which we may *define* as the magnetization term.[1] Having said this much about it, we may avoid all further detailed consideration of the electronic motions responsible for it; and, remembering only that its divergence is zero, we may proceed as follows:
Whatever a may be, we have as a purely mathematical theorem:

$$4\pi a=-\int\int\int\frac{\nabla^2 a}{r}d\tau-\int\int\frac{1}{r}\left[\left(\frac{\partial a}{\partial n}\right)_1-\left(\frac{\partial a}{\partial n}\right)_2\right]dS$$

where the value of a on the left hand side corresponds to the value at a point in the region outside of the boundary of the medium (1), and the triple integral extends throughout all space. In our case, a is the contribution to the vector potential by the magnetization. Since $Div\ a=0$, we have

$$\nabla^2 a=-Curl\ Curl\ a=-Curl\ m$$

[1] In the classical derivation of the formulae the $\Delta\dot{E}$ term does not appear to be considered. It does not show itself in the process of averaging the current density throughout the macroscopic volume element, although it should be made to do so. The percentage variations of E throughout the macroscopic volume element are of the same order of magnitude as those of the charge density ρ, and there is no more justification for neglecting their effects than there would be for saying that the electrostatic effect of a macroscopically neutral, but polarised volume element was zero because it contained equal quantities of positive and negative charge. Moreover, the services of the $\Delta\dot{E}$ term is required for the purpose of completing the magnetization term. Without it, this term would not give to the vector potential a contribution having zero divergence, and it would not correspond to what would be *measured* as the magnetization, even in the case of a non-moving medium.

where m is the value of the magnetization corresponding to the part a of the total vector potential. Thus,

$$4\pi a = \iiint \frac{Curl\ m}{r} d\tau - \iint \frac{1}{r}\left[\left(\frac{\partial a}{\partial n}\right)_1 - \left(\frac{\partial a}{\partial n}\right)_2\right] dS$$

Now it is easy to see that

$$-\left(\frac{\partial a}{\partial n}\right)_1 = [n_1 m_1],\ \left(\frac{\partial a}{\partial n}\right)_2 = -[n_2 m_2]$$

where n_1 and n_2 refer to the unit normals in the media (1) and (2) respectively. Thus, as far as the analytical contribution of the magnetization of the material medium to the vector potential is concerned, we have the equivalent of a distribution of current density of amount $c\ Curl\ m$ throughout the volume, and a surface current sheet of density $c[n_1 m_1]$ at the surface. In the case of a boundary between two material media it is the discontinuity of m which is involved.

Concerning the electrostatic potential ψ there is little to be said. The ordinary processes familiar in the development of Poisson's theory of dielectrics lead immediately to the fact that, as regards the polarization, the medium acts as though it were endowed with a fictitious volume charge of density $-\ Div\ P$ and a fictitious surface charge P_n per unit area.

Thus, to sum up, we have the following: Material media act as a whole as though there were

(1) a volume current density of amount

$$j + \rho u + \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} + Curl\ [Pu] + c\ Curl\ m$$

(2) a volume charge density equal to $\rho - Div\ P$

But this specification is not sufficient, for we must have also

(3) a surface current density equal to

$$u\sigma + u\overline{P}_n + c\ Curl\ [n\overline{M}]$$

(4) a surface charge density equal to $\sigma + \overline{P}_n$

where $\overline{M}$ and $\overline{P}_n$ refer respectively to the discontinuities in m and P_n at the boundaries, and where σ represents the real charge density.

Moreover, it is not quite correct to say that (1) is confined to the volume elements and (3) to the surface. Thus, in the foregoing presentation, the term corresponding to uP_n occurred fundamentally in the volume elements. Although it is not there represented as an influence which is expressible in terms of *any* volume density of current macroscopically specified, it is of such a nature that, on integration throughout the volume, it contributes a result which is expressible in terms of a surface current.

In Lorentz's treatment, but with his special meaning for the polarization, (1) is completely representative of the action of the volume element as regards current. On the other hand, the surface terms such as uP_n are not avoided. This particular term appears now, not as a residue from the volume elements, collected up as it were by Green's lemma and deposited on the surface, but as a surface current resulting from the motion of the charges left over at the surface in order to preserve constancy of the total electric charge in the medium in spite of the displacements which constitute the polarization. Analogous remarks apply to the surface term resulting from the magnetization.

It will be observed that there is a fundamental difference between the representation of phenomena in terms of a fine-grained analysis and the representation in terms of the macroscopic analysis. For while in the former, the general law is expressible entirely by the differential equations, other matters such as boundary conditions applying only to the special problem, in the macroscopic analysis, the differential equation is not sufficient, and the surface effects play a part as sharing in the expression of the general law.

Appendix

Note 1. Concerning definitions in terms of a unit pole or unit magnet.—(Page 12). Thus, to illustrate a few of the difficulties involved in an endeavour to maintain logical precision in this matter, it may be remarked that the position of the axis of the unit magnet requires definition in terms of experiment. It is useless to say that it is the line joining the poles, without stating more explicitly what we mean by the statement. Perhaps the best that can be done is to say that if two similar magnets be turned so as to exert the greatest attractive force upon each other for a given distance, the line joining their centers is the axis of each. A small unit magnet may be defined in terms of the force which it exerts upon another just like it, when the two are placed with axes in line. A logical difficulty becomes involved, however, from a consideration of the alteration in moment which each produces in the other when the two are in each other's proximity. It is not safe to relegate this effect to the realm of trivialities; for, without further examination of the theory of the phenomena, one is unable to say whether the said inductive action is or is not a logical necessity of the actual existence of the magnetic state in either. Neither can one say that he will avoid concerning himself with these matters by *defining* the unit magnet in the above way and then sticking to his definition by calling it a unit magnet even after he has removed it from the influence of its neighbour. Truly, nobody can prevent one from defining a quantity in any way he chooses; but, we can readily see that a definition of unit magnet on the above lines and the subsequent definition of fields in terms of it would lead to fields whose properties are not those of which we have been accustomed to speak. The best that we can do in order to retain approximate precision in this matter is to include in our definitions a statement of how the moment of our test magnet is to be corrected on account of the field in which it exists and whose purpose it is to measure.

Again, one may say that he can approximate to a unit pole by taking a very long thin magnet. It is to be observed, however, that the proof that such a magnet acts in this way is something involving all sorts of hypotheses. We must first assume that magnetism is explicable in terms of doublets or their equivalents (amperian whirls for example) and then, after developing the mathematics showing that the action of the magnet can be represented in terms of fictitious volume and surface charges, an analysis involving the assumption of the law of inverse squares, moreover, we require the further assumption that our magnet is uniformly polarized before we can say that it can be represented entirely by a surface distribution of magnetism. When we have done all this we can show that if it is very thin, the surface distributions approximate to point singularities at the ends. One's mind rather revolts at the thought of putting the said pole inside an electron for the purpose of measuring the magnetic field there. Yet, electron theory does not hesitate to concern itself with the field within an electron.

Note 2. Several writers have given formulations of the laws of electro-magnetism in accordance with dynamical principles.

Equations for free aether.—(Page 23). Equations exactly corresponding to the form of the circuital relations of electromagnetic theory as applied to free aether were obtained by MacCullagh[1] as early as 1839 although without thought of their relation to electrical phenomena. Starting with the assumption that the kinetic energy was of the form

$$T=\frac{1}{2}A\int\int\int(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)d\tau \tag{85}$$

where ξ, η, ζ, corresponds to displacement of the medium, and assuming a purely rotational type of elasticity, so that the potential energy is of the form

$$V=\frac{1}{2}B\int\int\int\left[\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right)^2+\left(\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}\right)^2\right]d\tau \tag{86}$$

MacCullagh by application of the Hamiltonian principle deduced the equations

$$-A(\ddot{\xi},\ \ddot{\eta},\ \ddot{\zeta},)=B\ Curl\ (f,\ g,\ h,) \tag{87}$$

where $(f,\ g,\ h)$, is defined as regards its rate of change by

$$(\dot{f},\dot{g},\dot{h},)=Curl\ (\dot{\xi},\ \dot{\eta},\ \dot{\zeta}) \tag{88}$$

Equations (87) and (88) correspond exactly to the electromagnetic equations for free aether if aetherial velocity $(\dot{\xi},\ \dot{\eta},\ \dot{\zeta})$ is taken to represent magnetic field, and $(f,\ g,\ h)$ representing curl of the aetherial displacement is taken to represent $\left(\frac{A}{B}\right)^{\frac{1}{2}}\times$ electric field. On this theory the kinetic energy is represented by

$$T=\frac{1}{2}\int\int\int(H_x^2+H_y^2+H_z^2)d\tau$$

Equations for media containing charges.—By utilizing the relation $H=Curl\ J$, where J has the meaning defined in equation 29, and by substituting in (89) and integrating by parts, the relation

[1] "An Essay Towards a Dynamical Theory of Crystalline Reflection and Refraction." Trans. R. I. A., Vol. 21, Collected Works, p. 145.

$$\frac{1}{2}\int\int\int(H_x^2+H_y^2+H_z^2)d\tau=\frac{1}{2c}\int\int\int\Big\{J_x(\rho v_x+\dot{E}_x)+J_y(\rho v_y+\dot{E}_y)$$
$$+J_z(\rho v_z+\dot{E}_z)\Big\}d\tau+\frac{1}{2}\int\int\Big\{(nH_y-mH_z)J_x \qquad (89)$$
$$+(lH_z-nH_x)J_y+(mH_x-lH_y)J_z\Big\}dS$$

is obtained, the surface integral being extended over a boundary surrounding the region throughout which the volume integrals are taken. It has been customary to assume that the surface integral vanishes when the surface is taken at infinity, and that the two expressions

$$T=\frac{1}{2}\int\int\int(H_x^2+H_y^2+H_z^2)d\tau \qquad (90)$$

and

$$T=\frac{1}{2c}\int\int\int\Big\{J_x(\rho v_x+\dot{E}_x)+J_y(\rho v_y+\dot{E}_y)+J_z(\rho v_z+\dot{E}_z)\Big\}d\tau \qquad (91)$$

are equivalent.[1] The second form for T is the one which is used in deriving the equations for a medium containing electrons. As pointed out by Macdonald,[2] the neglect of the surface integral is in general not legitmate, and the two forms of dynamical theories which take respectively equations (90) and (91) for the kinetic energy are not the analytical equivalents of each other.

In view of the fact that it is customary in many branches of electromagnetic theory, to take the former of the above expressions as the appropriate one for kinetic energy, Livens has attempted to obtain the equations for a medium containing electrons on this basis.[3] Livens writes[4]

$$T=\frac{1}{2}\int\int\int H^2d\tau \text{ and } V=\frac{1}{2}\int\int\int E^2d\tau$$

He then carries out the variation of the Lagrangian function

$$\int dt\int\int\int(H^2-E^2)d\tau$$

subject to four conditions of constraint, viz.,

$$Div\ E-\rho=0, \text{and } Curl\ H-\frac{1}{c}\frac{\partial E}{\partial t}-\rho\frac{v}{c}=0$$

The second equation being a vector equation really expresses three conditions.

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[1] See J. Larmor, "Aether and Matter," p. 92

[2] H. M. Macdonald, "Electric Waves," p. 33.

[3] G. H. Livens, "The Theory of Electricity," pp. 567–572.

[4] We here make no distinction between B and H since we are concerned only with pure aether and electricity, and are not dealing with a macroscopic analysis.

The first of the above conditions of constraint introduces the undetermined multiplier φ, which subsequently turns out to be the electrostatic potential, while the other three conditions introduce undetermined multipliers which turn out to be the three components of the Maxwellian vector potential.

Although this analysis appears at first sight to avoid the difficulty inherent in the use of (90) for the kinetic energy expression, it must be observed that a surface integral becomes thrown out by the integration by parts occurring in the process of carrying out the variation of the Lagrangian function, and this surface integral involves the vector potential J. It is necessary to assume therefore that J vanishes to a higher order than $1/r^2$. In other words, in addition to the ordinary equations, the dynamical analysis automatically imposes this condition upon the systems which it is to regard as falling within the scope of its correlation. If we adopt this condition, however, the surface integral in (89) no longer gives any trouble. In other words it would appear that the difficulty under discussion is only avoided by the limitation of the problem to those cases in which it does not exist.

Considerable interest attaches to the manner in which the force equation evolves in the various dynamical analyses. According to the plan of the analysis sketched on pages (26) to (28) of this report, it would appear that the force equation should really take the form $E+\frac{[vH]}{c}=o$ at each point, even at a point within the electron. On the other hand Larmor appears to obtain relations of the type[1]

$$m\ddot{r}=E+\frac{[vH]}{c} \tag{92}$$

where the vector $\ddot{r}$ represents the acceleration, and m is a constant depending upon the size and constitution of the electron. In view of the customary expression of the equations of motion of an electron in the form "External Force" = Mass$\times$acceleration, with the mass a function of the velocity, and also with the realization that even in this form the equation is only an approximation, it becomes of interest to trace the origin of (92).

The differential element of volume employed by Larmor is one containing a multitude of electrons. It is in fact a macroscopic volume element. Now, in building up the Lagrangian function, Larmor speaks of the "Part of the kinetic energy which involves the single electron moving with velocity $\dot{x}$, $\dot{y}$, $\dot{z}$," and he writes it in the form

$$\frac{1}{2}Le^2(\dot{x}^2+\dot{y}^2+\dot{z}^2)+e\frac{\dot{x}J_x}{c}+e\frac{\dot{y}J_y}{c}+e\frac{\dot{z}J_z}{c}$$

[1] "Aether and Matter," p. 97.

where L is a constant. In writing the expression in this way, it becomes implicitly involved, as will be shown, that the quantity J which occurs in the expression and is instrumental in determing the vectors E and H which occur in (92) is not the whole of the vector potential, but only the part thereof which remains when the main contributions from the electron itself have been subtracted.

Expressed in vectorial notation, equation (28) for the kinetic energy takes the form

$$2cT=\int\int\int(J.\overline{\rho v+\dot{E}})d\tau$$

Let us consider all the contributions to this quantity, in which some one electron figures. The result is given by ΔT, where

$$c\Delta T=\underset{(1)}{\int\int\int}\left(\left\{\rho v+\dot{E}\right\}d\tau\underset{(2)}{\int\int\int}\left\{\frac{\rho v+\dot{E}}{r}\right\}d\tau\right)$$

where the integrals labeled (2) are taken throughout all space but r is the distance from the volume element to a point inside the chosen electron, and where the integrals labeled (1) are taken over the electron. We may divide the integrals (2) into two parts, a part which corresponds to the integral over the space of the electron, and a part which we shall call J_0, which corresponds to the integration over the remainder of space. We thus obtain

$$c\Delta T=\int\int\int\left(\left\{\rho v+\dot{E}\right\}d\tau\cdot\int\int\int\left\{\frac{\rho v+\dot{E}}{r}\right\}d\tau\right)+\int\int\int\left(\left\{\rho v+\dot{E}\right\}\cdot J_0\right)d\tau$$

Here the first pair of triple integrals are taken over the volume of the electron alone, while the third triple integral is taken throughout the space. Dissecting the expression still further, we have

$$\begin{aligned}c\Delta T=&\left(\int\int\int\rho v d\tau\cdot\int\int\int\frac{\rho v}{r}d\tau\right)+\left(\int\int\int\dot{E}d\tau\cdot\int\int\int\frac{\dot{E}d\tau}{r}\right)\\&+\left(2\int\int\int\dot{E}d\tau\cdot\int\int\int\frac{\rho v}{r}d\tau\right)+\left(J_0\cdot\int\int\int\rho v d\tau\right)\qquad(93)\\&+\left(J_0\cdot\int\int\int\dot{E}d\tau\right)\end{aligned}$$

where the taking of J_0 outside of the integrals can only be justified on the approximate assumption that, in so far as it is made up of contributions from regions other than that of the electron itself, its variation over this region is negligible. If we now write $v=v_0+\Delta v$, where v_0 is the velocity of the center of the electron, and Δv is the departure from that value, (93) results in

$$
\begin{aligned}
c\Delta T=(v_0\cdot v_0)\iiint\rho d\tau\iiint\frac{\rho d\tau}{r}+\left(\iiint\rho\Delta v d\tau\iiint\frac{\rho\Delta v}{r}d\tau\right)\\
+\left(2v_0\iiint\rho d\tau\iiint\frac{\rho\Delta v}{r}d\tau\right)\\
+\cdot\left(\iiint\dot{E}d\tau\iiint\frac{\dot{E}}{r}d\tau\right)+\left(J_0\iiint\dot{E}d\tau\right)\\
+\left(2v_0\iiint\dot{E}d\tau\iiint\frac{\rho d\tau}{r}\right)+2\iiint\dot{E}d\tau\iiint\frac{\rho\Delta v}{r}d\tau\\
+\left(J_0\cdot v_0\right)e+\left(J_0\cdot\iiint\rho\Delta v d\tau\right)
\end{aligned}
\qquad (94)
$$

Now as regards the terms on the right hand side, the first is obviously of the form $L_0e^2(v\cdot v)$, where the quantity L_0 is a constant depending upon the shape and distribution of charge in the electron. This term, together with the eighth member of the right hand side when written out, is responsible for the whole of the expression

$$\frac{1}{2}Le^2(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{e\dot{x}J_x}{c}+\frac{e\dot{y}J_y}{c}+\frac{e\dot{z}J_z}{c}$$

which is taken by Larmor as contributing to the portion of the kinetic energy in which the electron in question figures. As regards the remaining terms of (94) it will be observed that if all parts of the electron move with the same velocity, some of the terms vanish, leaving, however, the terms

$$\left(\iiint\dot{E}d\tau\iiint\frac{\dot{E}}{r}d\tau\right)+(J_0\iiint\dot{E}d\tau)+2v_0\iiint\dot{E}d\tau\iiint\frac{\rho}{r}d\tau$$

It is these terms which, even in the case of a rigid electron would play the major role in importing into the equation of motion those characteristics which are ordinarily depicted by a variation of mass with velocity.

In the form of the analysis as presented by Macdonald, the application is to true differential elements of volume as distinct from differential elements containing a large number of electrons. Macdonald includes, however, in his original expression for the Lagrangian function a part which he writes as $\iiint L'dxdydz$, and which follows right through the analysis and appears in the equation of motion in the manner given by

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial\dot{x}}\right)-\frac{\partial L'}{\partial x}-\rho\left(E_x+\frac{[vH]_x}{c}\right)=0$$

If L' were of the simple form $\rho_0\dot{r}^2/2$, this extra part of the Lagrangian function would of course simply result in

$$\rho_0\ddot{r}=E+\frac{[vH]}{c}$$

Although expressed in a somewhat different analytical form, the inclusion of a part L' in the Lagrangian function is the equivalent of the method adopted by H. A. Lorentz.[1]

In concluding these remarks upon the dynamical methods applicable to the subject it may be well to call attention to one additional matter. None of the dynamical developments appear to take account of that outstanding feature of the subject embodied in the fact that all electrons have the same charge. Would it not be possible to develop a more complete analysis in which this fact was taken into account in the form of an extra condition of constraint?

Note 3. Concerning the significance of v in the circuital relations.—(Page 25).—We may speak of the velocity of a point which is postulated to have some means of identification; but, to speak of the velocity in a continuous distribution of electricity is meaningless without further definition. When we speak of the velocity of a stream of water, we really mean the velocity of a cork in the stream or the average velocity of a molecule of which the stream is composed. Both of these definitions become without meaning as applied to the velocity of the electricity within an electron. If we are speaking of the velocity of a stream of electrons the situation is one of less perplexity; but the electromagnetic equations claim the power of applicability on an infinitely small scale, and, in some of their most important applications to the theory of electrons, the interrelation of the fields and motions in different parts of the same electron are matters of fundamental moment.[2] It is naturally of the greatest importance to understand what is to be meant by v in a case of this kind. Here again the question is largely one of the light in which we regard the definitions of the other quantities. We have already enlarged upon the difficulties in making logical definitions of E and H; and, without taking any responsibility for the definition of these quantities have set ourselves the task of inquiring how matters stand provided that we start from the hypothesis that there *exists* some logical definitions for them. Following out this line of thought, it would seem that equation (17) may be regarded in the following light.

E and H being assigned, define a quantity I by the relation

$$I = c \, Curl \, H - \dot{E} \tag{95}$$

where c is a perfectly arbitrary constant supposed at our disposal. It is of course only after c is assigned that our definition becomes explicit. Define v by the relation

$$v = \frac{I}{Div \, E} = \frac{I}{\rho} \tag{96}$$

[1] "La Theorie Electromagnetique de Maxwell" "Archives Neerlandes des Sciences," v. 25, 1891–92.

[2] In calculating the so called "force which the electron exerts upon itself as a result of its motion," for example.

so that from (95) and (96),

$$\frac{1}{c}(\rho v+\dot{E})=Curl\ H \tag{97}$$

Now as regards the above definitions, nobody can prevent our defining I and v in any way we please. We cannot discover any laws of nature by definition, however; and, in general, the quantity v in (97) will have a very artificial meaning unless we make some further assumptions. To illustrate this, suppose we consider a spherical region (an electron for example) whose boundary is determined by the condition that *Div E* suffers a sharp change. Suppose, moreover, that this boundary is at rest. We might be inclined to say that the average value of v throughout the interior of the boundary is zero. How do we know, however, that electricity is not running in at the left side with infinite velocity, and out at the right hand side with infinite velocity, so that while the density outside of the electron is zero, ρv is finite there? Moreover, if we suppose a condition where the spherical boundary, as determined by the place where *Div E* suffers a sharp change, moves with a velocity u, then, quite apart from all question of relative motion of electricity within the electron itself, we are not sure that ρu represents the value of ρv which should occur in (97). For, we can imagine superposed on ρu a condition in which electricity runs into the boundary at the left and out of it at the right, with infinite velocity in such a manner as to contribute anything we like to ρv. It must be remarked that the above discussion in terms of infinitely small densities and infinite velocities, and the rather inexact use of words involved in speaking of them is only introduced here by way of illustration. In view of our purely artificial definition of I, it is conceivable that this quantity might have nothing whatever to do with the motion of electricity; in this case, however, our act of dividing it by a ρ which might have no existence, and so defining a v which when multiplied by the said ρ reproduced I would be obviously nothing more than a manipulation of symbols. Our desire is to emphasize the fact that, until some further assumption is made, there is no connection whatever between the velocity of the boundary of our charge and the v which occurs in equation (97). Co-existent with any set of motions which we like to assign to the boundaries of the charges, I, i. e., ρv can have any values whatever, as far as our definitions are concerned, and this is what might be expected in view of the fact that equation (97) figures merely as the definition of v. How then are we to import reality into the v of equation (97)?

Let us integrate the equation over a long period T

$$\int_0^T \rho v dt+\int_0^T \dot{E}dt=c\int_0^T Curl\ H\ dt \tag{98}$$

As in the above discussion, let us restrict ourselves for the time being to a case where ρ exists in the form of discrete particles (electrons), determined as regards their boundaries by the condition that *Div E* suffers a sharp change there. By taking the velocity which we could conceivably measure with a meter stick for any point on the boundary of one of the particles, we may, in the case where that velocity is in the same direction for some considerable time, speak of it as the velocity of the electron as a whole with an uncertainty of meaning which is, in the limit, infinitesimal, even though the electron changes in shape while in motion.[1] Let us, moreover, consider a case where the time integral of $\partial E/\partial t$ is negligible compared with that of ρv or of *Curl H*, so that we may write

$$\int_0^T \rho v dt = c\int_0^T Curl\ H\ dt \tag{99}$$

Let us further integrate with respect to an element of area *ds* over an area of 1 sq. cm.

$$\iint ds \int_0^T \rho v dt = c \iint ds \int_0^T Curl\ H\ dt \tag{100}$$

Without making any assumptions about the rigidity of an electron, let us consider an electron in the process of passing through the plane

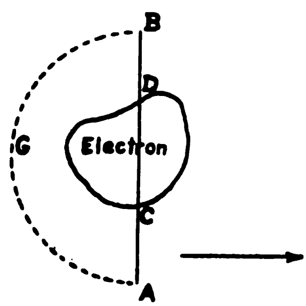

FIG. 2.

AB, Fig. 2. As a direct analytical consequence of equations (17) and (19), and without even the necesssity of any detailed meaning to v as the velocity of electricity, it follows that

$$-\frac{\partial \rho}{\partial t} = Div(\rho v) \tag{101}$$

[1] In case one should argue that he has no means of fixing a point on the surface of the electron, we may suppose the velocity determined by taking the first reading for any one point, and the second reading for any other point. The difference of the readings divided by the time will, in the case of a long time interval, give the velocity with infinitesimal uncertainty as to its meaning.

and consequently

$$-\frac{\partial}{\partial t}\iiint \rho d\tau = -\iiint \frac{\partial \rho}{\partial t} d\tau = \iiint Div(\rho v) d\tau \qquad (102)$$

the integrals being taken throughout any volume whatever. Applying Green's lemma[1] to the last member of (102), we have

$$-\frac{\partial}{\partial t}\iiint \rho d\tau = \iint \rho v_n ds = \frac{\partial}{\partial t}\iint ds \int_0^T \rho v_n dt \qquad (103)$$

where the surface integral is taken over the surface surrounding the volume. Let us take our surface as composed of a portion AB large enough to include the greatest cross section of the electron as it passes through, and a surface such as the dotted surface, so that our complete surface of integration is ACDBGA.

Now it will be recalled that, in view of the artificial origin of ρv, a part of this quantity may represent a portion of I which has really no status in terms of motion of electricity, and which, by being forced into this form, calls for what would be described in physical terms as a continual flow of electricity *through* the moving electron, the velocity being infinite and the density zero at entrance and exit. If we wish to *postulate* that ρv_n is zero at all points where ρ is zero, the integral on the extreme right of (103) need only be extended over the plane AB, since the part over BGA will not contribute anything. Moreover, integrating over a time corresponding to that of the passage of the electron across the plane, we have, for the contribution of this electron to the integral on the extreme left hand side of (103) a value equal to the integral of $\rho d\tau$ throughout its volume, and this integral constitutes the definition of the charge on the electron. If N electrons pass through one square centimeter in the time T, the total value of the integral on the left hand side of (103) is Ne, so that, in view of (100),

$$Ne = c\iint ds \int_0^T Curl\ H\ dt$$

where, in accordance with the above, the surface integral is taken over one square centimeter of the plane AB.

The assumption which has been made amounts to this, that it is possible to choose a constant c such that, with I defined as in (95), the whole of this quantity I may be represented as regards its time

[1] In order to apply Green's lemma in this form we must strictly have no discontinuity in $Div\ \rho v$. This difficulty is really present in most of the classical demonstrations where Green's lemma is applied to charges with sharp boundaries. It can be mathematically continuous, but as sharp as we please. With this understanding, our equation (103) holds.

integral over a long period, by the transference of the charges on the electrons. Since

$$\int\int ds \int_0^T \mathit{Curl}\ H\ dt$$

is what is defined as the Heavisidean electromagnetic measure of the quantity of electricity passing through the square centimeter in the time T, while Ne is the Heavisidean electrostatic measure of this quantity, the ratio of the electromagnetic unit to the electrostatic unit is equal to c.[1]

It is important to realize that the element of *assumption* which has been here introduced, has been brought in not as a necessary part of the electromagnetic equations themselves, but rather in connection with their application to the particular problem of a stream of electrons. Equation (97), as it stands, involves no assumption; it is merely the definition of v, and is the only meaning to be attached to v at a *point* as distinct from the velocity of an electron as a whole. For the purpose of definition, c could be anything. The only place where assumption comes in is in discussing the contribution of those moving identifiable particles which we call electrons, and which we suppose to be characterized by showing a discontinuity, or at any rate a recognizably sharp change in ρ. It becomes of convenience to import into the definition implied in equation (97), the value of c which is chosen so as to fit the special problem in which assumption is involved, so that by the choice of this c, the v which is implicitly defined by (97) shall revert for that special case to its proper relationship with the meter-stick velocity of the electron as a whole.

Speaking in terms of physical intuition, but perhaps in terms which are not very exact, we may say that the significance of c is this: We may choose c to have any value we like, but then, in the case of a stream of electrons of constant velocity it will be necessary to regard the velocity of the electron as a whole as different from the average velocity of the charge in it. In other words, as we have before remarked, charge will flow in at one side of the electron and out at the other as the electron goes along. For any value of the velocity of the electron we can always choose a c such that the average v shall represent the meter-stick velocity of the electron; but, it is an assumption that the same c will serve in this respect for all velocities.

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[1] The exact specification of the conditions involved in speaking of the ratio of the electrostatic to the electromagnetic unit must necessarily involve a rather cumbersome presentation, if we use the electromagnetic unit only in the sense in which it is defined, viz., as the time integral of a current which is itself defined in terms of the curl of a magnetic field.

Note 4. Concerning definitions in terms of the motion of an electron.— (Page 31). If it be our intention to make definitions of that type which speaks of force on unit charge, unit pole, etc., the limit of refinement is reached when we choose as our entity on which to discuss the force the smallest particle whose existence we are willing to admit. Choosing then the electron as the entity in terms of which to construct our definitions, let us define the electric field E by the relation

$$E=k(\ddot{s})_{v=0} \tag{104}$$

where $(\ddot{s})_{v=0}$ is the acceleration which the electron would experience if it existed at the point in question with zero velocity, and, we should strictly add, with every time derivative of its coordinates, other than the acceleration, zero also. The constant k is to be regarded as perfectly arbitrary, and we shall find it possible to leave it so throughout, in such a manner as to avoid the difficulties accompanying the introduction of a material mass, the realization of a unit charge, etc.

As regards H, it will be realized that the fundamental physical idea involved in the usual definition in terms of the couple on a magnet, or current circuit, is the existence of a mechanical force on a current element. In its electronic aspect this reverts to the force on a moving electron. Using this as a suggestion to guide us, it might appear appropriate to *define* the component of H parallel to the axis of z for example by the equation[1]

$$k(\ddot{\xi})_{v\neq 0}=E_x+\frac{(v)_y H_z}{c}$$

where $(\ddot{\xi})$ represents the acceleration of the electron along the x axis, when the only velocity which it has is the velocity $(v)_y$ along the y axis. Recalling that $E_x=k(\ddot{\xi})_{v=0}$, the above would become

$$k(\ddot{\xi})_{v\neq 0}-k(\ddot{\xi})_{v=0}=(v)_y H_z/c \tag{105}$$

with corresponding definitions for H_x and H_y. As a matter of fact, we shall elaborate this definition slightly by expressing it in a form as typified by H_z as follows:

$$\frac{H_z}{c}=\underset{v=0}{\mathrm{L}^t}\ \frac{k(\ddot{\xi})_{v\neq 0}-k(\ddot{\xi})_{v=0}}{(v)_y} \tag{106}$$

The reason for defining H in terms of (106) rather than in terms of (105) is this: We know that, in the language of ordinary electromagnetic theory, the mass varies with the velocity. In terms of the ordinary assumptions which occur in that theory, where (105) figures as the equation of motion of the electron, the quantity k on the left hand side of (104) would be multiplied by some function of the velocity. In

[1] It will at once be observed that this relation is suggested by $F=E+[vH]/c$. It would not, of course, be proper to attempt to define the magnetic field by supposing the electric field absent.

other words, the acceleration $(\ddot{\xi})_{v\neq 0}$ is not independent of v. It depends only on v^2/c^2, however, whatever type of electron be assumed, so that H_z as defined by (106) approaches a definite limit which is independent of the different assumptions involved in the different electrons. Of course, nobody can prevent our *defining* H in any way we choose, for example, without the limiting condition specified in (106). If we do this, however, it will in general result that, in terms of the H we have so defined, we shall find equation (18) to be untrue.

The ultimate form assumed by the circuital relations.—If, in terms of the definitions given by equations (104) and (105) we substitute in equation (18), the constant k cancels throughout, leaving the result in the form

$$-\underset{(v_y=0)}{Lt}\frac{\frac{\partial}{\partial t}(\ddot{\xi})_{v\neq 0}-\frac{\partial}{\partial t}(\ddot{\xi})_{v=0}}{(v)_y}=\left(\frac{\partial\ddot{\eta}}{\partial x}\right)_{v=0}-\left(\frac{\partial\ddot{\xi}}{\partial y}\right)_{v=0}$$

with two other relations corresponding to the other two components. This involves nothing more than a statement of a relation between the derivatives of the motion of an electron at the point under different conditions. Not a single constant, not even c, is left.

On the basis of definitions which give E and H in terms of the motion of an electron as a whole, equation (17) can hardly be regarded as having a meaning except as applied in free aether, or in the hypothetical case of a non-electronic, and more or less continuous distribution of electricity. Viewed in this sense, however, and remembering that ρ is defined as *Div* E, we see again that when the vectors E and H are replaced by the quantities which define them, k cancels throughout, leaving an equation which expresses a relation between v and the derivatives of the motion of the electron at a point under different conditions.

With regard to this cancellation of k, the point is that although, to the extent that k is arbitrary, H is arbitrary, to that extent E is also arbitrary, and ρ is arbitrary; and, the relations between E, H, ρ, and v for one value of k are exactly the same as those between the corresponding quantities for any other value of k. In fact, the difficulty attending the conception of an ordinary material mass and a unit of charge in our definitions has disappeared as indeed it ought to. For, nobody ever observes a mass directly in the case of an electron . He writes down an equation with a mass in it, then does some experiments, calculates the mass, and finally puts it in a table in order that somebody else may insert it in his equations for the purpose of calculating how *his* observable electrons move. If we concern ourselves only with what is actually observed, we need not introduce the ideas of measure of charge or mass at all. Everything can be formulated in terms of a statement about how the time derivatives of the motion of the electron or other observable entity are related to each other under different conditions.

It must be remarked that the definition (106) does not constitute the complete equation of motion of the electron, since it is in the form of a limiting relation where the velocity is zero. This is in line with what might be expected, since we know that the complete calculation of the equation of motion of an electron involves some other assumption, such as that of Lorentz for example, which, in the ordinary language of electromagnetic theory gives the law in the form that the electron moves in such a manner that the total force which it exerts on itself due to its motion is equal and opposite to the force on it due to the external field.

Relation between the above definition of H and that ordinarily employed.—In the more coarse grained specifications employed in practise, it would appear that the most logical procedure is to first obtain the currents in the circuits in terms of the forces between the circuits in the manner outlined on pages 13-15. On this basis, the definition of H given by (8) is, in view of (9), the analytical equivalent of that which defines it at a point in terms of the mechanical force on the element of wire carrying the current at that point. As already pointed out, this method is applicable only in the case where the currents are steady or vary very slowly.

Now our equation (106) defines a quantity H which, in the case of an electron moving with small velocity results in an acceleration proportional to that velocity. If we assume that this action becomes transmitted to the wire, and finally to the spring which holds the coil, in the form of a pull proportional to the sum of accelerations which the electrons would suffer if free, the quantity H which we have defined by (106) falls into line with the more coarsely defined quantity referred to above, in the only case where that coarse-grained definition can be experimentally realized. Having defined the unit H in the coarse-grained way, in terms of our ordinary definitions, if we should now put a moving electron into the field and observe its acceleration, the constant k would of course become determined as the special value which caused the H defined by (106) to be numerically equal to the unit which is defined for the coarse-grained specification. But, for the purpose of the electromagnetic equations themselves, and for everything that they have to say about all that we can observe as regards the motions of electrons and so forth, correlations with the coarse-grained phenomena, with their attending approximations and vaguenesses of expression are unnecessary.

Note 5. The role played by definition, and by the force equation in relation to the invariance of the circuital relations under the transformation of the restricted theory of relativity.—(Page 35). It will be recalled that the application of the relativity transformation to the electromagnetic equations leads to a set of similar form but with

$$E_x,\ E_y,\ E_z,\ H_x,\ H_y,\ H_z,\ u_x,\ u_y,\ u_z,\ \rho$$

replaced by

$$E_x,\ \beta(E_y-\frac{v}{c}H_z),\ \beta(E_z+\frac{v}{c}H_y),\ H_x,\ \beta(H_y+\frac{v}{c}E_z),\ \beta(H_z-\frac{v}{c}E_y) \qquad (107)$$

$$\frac{u_x-v}{1-\frac{u_xv}{c^2}},\quad \frac{u_y}{\beta\left(1-\frac{u_xv}{c^2}\right)},\quad \frac{u_z}{\beta\left(1-\frac{u_xv}{c^2}\right)},\quad \beta\left(1-\frac{u_xv}{c^2}\right)\rho$$

$$\text{where } \beta=\left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

Replacing the second set of ten quantities by the first set of ten letters, but with dashes, it is customary to say that the equations have reverted to the same form and are in accordance therefore with the restricted theory. This is really more than one is justified in saying until he has shown that the dashed quantities are really the quantities which the moving observer will measure for the fields, velocities and densities. The result follows for the velocities directly from the transformation equations themselves, but we cannot say anything about the fields until we have discussed the matter in terms of the definition of those quantities; and, it would seem that in the usual treatment of the subject, this question is dismissed too lightly. If we are to define the quantity E in the manner discussed in Note 4, i.e. in terms of the acceleration of an electron, we shall have E_y, for example, defined by the relation.

$$E_y=m_0\left(\frac{d^2y}{dt^2}\right)_0$$

where m_0 is an arbitrarily chosen number, and where the zero subscript against the parenthesis is to indicate that the electron considered has zero velocity, and, we must add, zero rate of change of acceleration and of all higher derivatives; for, without such specifications, as to the higher derivatives, the definition would be ambiguous. Now it may be shown directly from the relativity transformation that an electron which in the dashed system, has zero velocity, and acceleration $\ddot{y}'_0$ has, in the undashed system acceleration $\ddot{y}_v$, given by

$$\ddot{y}'_0=\beta^2\ddot{y}_v$$

Hence, if we postulate that the electromagnetic equations are invariant under the relativity transformation, we shall require the condition

$$\beta\left(E_y-\frac{v}{c}H_z\right)=m_0\beta^2\ddot{y}_v$$

Analagous remarks apply in the case of the transformation of the other components of E. The three equations corresponding to the three components are included in the general vector equation (49), which consequently represents a requirement imposed upon the equation of motion of the electron in order that the *circuital relations* may be regarded as invariant under the relativity transformation. It is customary to regard the invariance of the circuital relations as established before the force equation is discussed.

As far as the mere requirements of relativity are concerned, equation (49), since it represents an invariant relation,[1] could be true for the most general type of motion. It is important to realize, however, that for the purpose of providing for the invariance of the circuital relations, it is only *required* for the special case of an electron which has velocity, acceleration, and no higher derivatives.

It remains to discuss how H must be defined in order that the invariance of the circuital relations shall be provided for. To define it as the force on a unit pole, would lead us into untold troubles, including the variation of the mass of the unit pole with velocity. The only logical way of proceeding in line with the plan adopted in the case of E is to define H in the manner outlined in Note 4, i. e., by utilizing the force on the electron in a direction perpendicular to its velocity.

We may, in fact, crystallize the whole situation as regards the definition of E and H in the following manner:

As already remarked, it is a known fact that if we write down the relation

$$E+\frac{[uH]}{c}=\frac{d}{dt}\frac{ku}{\left(1-\frac{u_x^2+u_y^2+u_z^2}{c^2}\right)^{1/2}} \tag{108}$$

an application of the relativity transformation will lead to

$$E'+\frac{[u'H']}{c}=\frac{d}{dt'}\frac{ku'}{\left(1-\frac{u_x'^2+u_y'^2+u_z'^2}{c^2}\right)^{1/2}} \tag{109}$$

where x', y', z', t', u', refer to the transformed system, and E', and H', have the significance implied in (107). Now we may use (108) to define E and H in the following way: When the velocity of the electron is zero, (108) leads to

$$E=k(\dot{u})_{u=0}$$

which we may take as the definition of E, in line with the method adopted in Note 4. By putting $u_x'^2+u_y'^2+u_z'^2=0$, and consequently $u'=0$ in equation (109), we see that the same definition in the dashed system makes E' the electric field in that system.

[1] The relation is invariant since, on multiplying throughout by $\left(1-\frac{u_x^2+u_y^2+u_z^2}{c^2}\right)^{-\frac{1}{2}}$ each side becomes converted into an expression capable of representing three of the four components of a 4-vector.

Again, if we write down the x component of equation (108) for the case where the velocity is entirely in the y direction, for example, we obtain

$$E_x + \frac{u_y H_z}{c} = \frac{d}{dt}\left[\frac{ku_x}{\left(1 - \frac{u_x^2 + u_y^2 + u_z^2}{c^2}\right)^{1/2}}\right]_{u_x = u_z = 0}$$

If we then replace E_x by $k(\dot{u}_x)_{u=0}$
and define H_z by the relation

$$\frac{H_z}{c} = \underset{u_y=0}{\mathrm{L}t}\, \frac{1}{u_y}\left[\frac{d}{dt}\frac{ku_x}{\left(1 - \frac{u_x^2 + u_y^2 + u_z^2}{c^2}\right)^{1/2}} - k(\dot{u}_x)_{u_y=0}\right]_{u_x = u_z = 0} \tag{110}$$

we shall have a definition which owing to the invariance in form of (108), defines H' in relation to the motions in the dashed system in the same way as H is defined in relation to the motions in the undashed system. In other words, the definition is one which permits of H' being regarded as the magnetic field in the dashed system.

By carrying out the limiting process as far as possible, the definition assumes the force

$$\frac{H_z}{c} = \underset{u_y=0}{\mathrm{L}t}\, \frac{k(\dot{u}_x)_{u_y \neq 0} - k(\dot{u}_x)_{u_y=0}}{u_y} \tag{111}$$

which corresponds to the definition (106) in Note 4. Similar remarks apply of course to the other components.

Case where E and H are not defined in terms of the motion of an electron.—If we wish to retain such a fine-grained specification of the equations that we may speak of the field at a point inside of an electron, definitions of the fields in terms of the motions of the electron as a whole are naturally too coarse-grained. It would now appear that the only logical procedure would be to follow the path outlined on pages (28) and (29), where E and H appear as subordinate quantities defined by equations (39) and (40), φ and U being in turn defined by (38), where the quantities ρ and v (which we shall here replace by u, to avoid confusion with the velocity of the system as a whole) are supposed determined on the basis outlined in the discussion of equations (36) and (37).

If we suppose the law typified by equations (36) and (37) to be restricted by the condition that it is invariant under the relativity transformation when that transformation is made subject to the condition that $\rho d\tau$ is invariant,[1] and that u transforms as a velocity, we shall have as follows:

[1] An interesting case of this is where the law concerns itself entirely with the description of the motions of particles, and where the density simply represents the number of particles per c.c., and where any constants associated with the particles for the purpose of the description are invariant under the transformation.

Suppose that in terms of the invariant distributions $\rho d\tau$, two observers, A and A'', the former using x, y, z, t, and the latter using x', y', z', t', make corresponding definitions of φ and U and of φ'' and U''. Suppose that they then make, in the form of equations (39) and (40), but in their respective coordinate systems, corresponding definitions of E, H, and E'', H''. Then, by the very meaning of these quantities, E'' and H'' are the quantities which play the parts of the electric and magnetic fields for the observer who measures x', y', z', t'. It is, however, well known, and follows as an immediate consequence of the expressions (107), that

$$\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right) dxdydzdt=\left(\frac{\partial E'_x}{\partial x'}+\frac{\partial E'_y}{\partial y'}+\frac{\partial E'_z}{\partial z'}\right) dx'dy'dz'dt'$$

Hence, the quantity $\rho' dx'dy'dz'dt'$, with ρ' defined as *Div* E' is the same as the quantity $\rho dxdydzdt$ used by the observer A in the definition of φ and U, and consequently of E and H, and is therefore the same as $\rho'' dx'dy'dz'dt'$ used by the observer A''. Hence $\rho''=\rho'$. Since, however, relations of the form (38)–(40) determine uniquely the values of E and H for given values of ρ and u, and since it results directly from the equations of transformation that u' is the quantity which A'' measures for the velocity of a particle, it follows that the vectors E' and H' are everywhere equal to E'' and H'', and consequently represent the quantities which observer A'' would regard as the electric and magnetic fields.

It is interesting to observe that the sole criterion for our being able to define a set of relations which are invariant under the relativity transformation in the sense in which that statement has meaning, and which are of the form of the electromagnetic circuital relations is the following: That it shall be possible to formulate the laws of nature in terms of the motions of elements of volume in the manner sketched on page 32, the numbers N there referred to depicting some physical quantity such as charge within the element of volume, and that it shall be possible to do this in a form such that all observers using the same law shall arrive at the same values for the numbers in the elements of volume. The *usefulness*, or *fundamentality* of the relations will still depend upon whether or not the motions of a volume element can be described in terms of the vectors E and H at the point without explicit introduction of the motions and numbers attached to the other volume elements in space.

Note 6. (Page 43). *Calculation of the term*

$$W=\int\int\int \Delta v\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau$$

In the first place it is to be observed that the integral may be replaced by $\int\int\int \Delta v\; E_1 \rho d\tau$, since Δv and $[vH]$ are everywhere perpendicular to each other. If the velocity is in the positive direction of the axis of x,

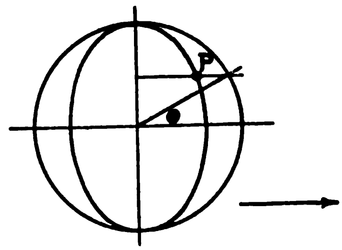

FIG. 3.

and if ξ is the perpendicular distance of a point P of the surface of the electron from the plane passing through the center of the electron and parallel to the yz plane, we have, for the Lorentzian electron,

$$\frac{\xi}{a \cos \theta} = \left(1 - \frac{v_0^2}{c^2}\right)^{\frac{1}{2}}$$

and

$$\Delta v = \dot{\xi} = -\left(1 - \frac{v_0^2}{c^2}\right)^{-\frac{1}{2}} \frac{v_0}{c^2} \cdot \dot{v}_0 \cdot a \, Cos \, \theta$$

Now, in the case of the Lorentzian electron, the elements of charge on the ellipsoid are obtained by projecting thereon, in a direction parallel to the axis of x, the corresponding elements of charge on the sphere which represents the electron in its un-contracted state. Thus, the contribution to the integral arising from the element of charge obtained by projecting the annular portion $2\pi a^2 Sin \, \theta d\theta$ of the sphere is

$$\Delta W = -\left(1 - \frac{v_0^2}{c^2}\right)^{-\frac{1}{2}} \frac{v_0 \cdot \dot{v}_0}{c^2} \cdot 2\pi a^3 \sigma \, Sin \, \theta \, Cos \, \theta \cdot E_1 d\theta \qquad (112)$$

where $\sigma = e/4\pi a^2$ is the surface density of charge on the sphere.

Now the field E_1 consists of two parts, a part corresponding to the unaccelerated motion of the electron, and a part corresponding to the acceleration. The latter part contributes, to the integral, factors proportional to $\dot{v}_0$ and higher derivatives, so that, in virtue of the $\dot{v}_0$ terms already in (112), these terms give rise to contributions proportional to the square of the acceleration, or to contributions involving the product of the acceleration and higher time derivatives. The whole of the discussion on pages (41) to (43) is founded upon the acceptance of the quasi-stationary principle as a sufficient approximation, and to this degree of approximation we need not therefore concern ourselves with these terms, since the quasi-stationary principle cannot be expected to include this approximation. These terms contain within them the phenomena of radiation, which is neglected in the present discussion, but which has nothing to do with the difficulty immediately under consideration. Confining ourselves therefore in the present case to the portion of the field which does not involve the acceleration and higher derivatives, we recall that since the x components of the fields of the sphere and ellipsoid are the same at corresponding points, the value of E_x at the point P just outside the ellipsoid is $e\, Cos\, \theta/4\pi a^2$. Just as, in the case of the system at rest, the average value of the field at a point within the charge on the surface is one half the value just outside, so here, as regards the integration across the thin element of charge at the point P, the effect is the same as if the magnitude of that element were multiplied by one half of the field just outside the ellipsoid at the point P. The value of W thus becomes

$$W = -\frac{e^2}{16\pi a}\left(1-\frac{v_0^2}{c^2}\right)\frac{v_0\dot{v}_0}{c^2}\int_0^\pi Sin\,\theta\, Cos^2\,\theta d\theta$$

$$= -\frac{e^2}{24\pi a}\left(1-\frac{v_0^2}{c^2}\right)\frac{v_0\dot{v}_0}{c^2}$$

Note 7. (Page 47).—This would immediately suggest that it would be unlikely that we should find a suitable general relation between P and E without introducing, for example, the quantity B. To illustrate this matter we may remark that if we write $\epsilon=(1-q^2/c^2)^{-\frac{1}{2}}$, where q is the resultant velocity of a point, then it is known that, in the usual notation,[1]

$$\epsilon\left\{E+\frac{[uB]}{c},\ i\frac{(Eu)}{c}\right\}$$

$$\epsilon\left\{E-\frac{[uE]}{c},\ i\frac{(Bu)}{c}\right\}$$

$$\epsilon\left\{D+\frac{[uh]}{c},\ i\frac{(Du)}{c}\right\}$$

$$\epsilon\left\{h-\frac{[uD]}{c},\ i\frac{(hu)}{c}\right\}$$

are 4-vectors. Thus, one possible pair of relations between D, E, B, and H consistent with relativity would be

$$\left\{D+\frac{[uh]}{c},\ i\frac{(Du)}{c}\right\}=K\left\{E+\frac{[uB]}{c},\ i\frac{(Eu)}{c}\right\};\ \left\{B-\frac{[uh]}{c},\ i\frac{(Du)}{c}\right\}$$

$$=\mu\left\{h-\frac{[uD]}{c},\ i\frac{(hu)}{c}\right\}$$

where K and μ are constants. When u is small, these reduce to the well known approximations $D=KE$, and $B=\mu h$.

[1] See, for example, E. Cunningham, "The Principle of Relativity," p. 119.

PART II.

UNIPOLAR INDUCTION

By JOHN T. TATE

I.

INTRODUCTION.

In his "Experimental Researches" Faraday[1] describes an experiment in which a current of electricity is induced in a wire whose ends are pressed against the surface of a cylindrical bar magnet spinning about its axis of symmetry. This type of induction was called "unipolar" by W. Weber[2] who analyzed experiments of this kind from the point of view of his two-fluid theory of magnetism. He regarded the induction as produced by the continuous passage of one pole only of a magnet through the current loop, the other pole remaining outside.

Today the term unipolar, acyclic, or homopolar is applied to the induction of an electric field in the vicinity of an axially symmetrical magnetic system which is rotating about its axis of symmetry. This magnetic system may be of any kind, permanent magnet, simple solenoid, or electromagnet. The problem of unipolar inducton has been correctly to account for the existence of this electric field. There seems always to have been an element of mystery connected with the phenomenon, and an element of doubt on the part of many physicists as to the power of electromagnetic theory to solve the problem uniquely.

It is proposed here to discuss the typical experiments on unipolar induction and their relation to modern electrical theory, to determine in how far the theory is in a position to predict the results of these experiments, and to what extent the results go beyond the theory in giving us information not already contained therein.

In the history of the development of the subject there has been a singular freedom from differences of opinion as to the experimental results, but at the same time a singular lack of agreement as to the way these results were to be interpreted. For that reason our attention in what follows will be directed primarily to the points of view of the different investigators rather than to details of experimental procedure. It will clarify the discussion, no doubt, if we first of all set forth briefly what present electromagnetic theory has to say about the unipolar induction problem.

[1] Faraday, Experimental Researches, v. 1, p. 63.
[2] W. Weber, Pogg. Ann., 42, 1841.

II

Electrodynamical Theory of Unipolar Induction

A. *Fundamentals.*—The modern theory of electromagnetism proposes to calculate all effects from the positions and motions of charges. The attempt is made to correlate the motions of these charges in such a way that, given their positions and motions at any particular instant, it will be possible to predict the state of things at some later time. The mathematical framework about which the theory is constructed is usually put in the form of the field equations:

$$\begin{aligned} \rho u+\dot{E}&=c\ curl\ H \\ \dot{H}&=-c\ curl\ E \\ div\ E&=\rho \\ div\ H&=0 \end{aligned} \tag{1}$$

together with the so-called "force" equation:

$$F=E+\frac{[vH]}{c} \tag{2}$$

It will be unnecessary here to say anything about the significance of these relations or of the quantities entering into them. These matters are fully discussed in Part I of this report.

It is customary to integrate the above equations in terms of a scalar potential, ϕ, and a vector potential, A, where

$$\begin{aligned} \phi&=\frac{1}{4\pi}\int\int\int\frac{[\rho]}{r}d\tau \\ A&=\frac{1}{4\pi c}\int\int\int\frac{[\rho u]}{r}d\tau \end{aligned} \tag{3}$$

These potentials are subject to the restriction, equivalent to the equation of continuity for the charge, that

$$div\,A=-\frac{1}{c}\frac{\partial\phi}{\partial t} \tag{4}$$

In terms of these potentials, E and H are given by:

$$\begin{aligned} E&=-\frac{1}{c}\frac{\partial A}{\partial t}-grad\ \phi \\ H&=curl\ A \end{aligned} \tag{5}$$

The square brackets in (3) have the usual significance of indicating that in evaluating the integrals we are to take the values of ρ and of ρu at a time $\frac{r}{c}$ earlier than the time at which we wish the values of ϕ and A.

Substituting in (2) we have:

$$F=-\frac{1}{c}\frac{\partial A}{\partial t}-grad\ \phi+\frac{[vH]}{c}$$

or

$$F=-\frac{1}{4\pi c^2}\frac{\partial}{\partial t}\int\int\int\frac{[\rho u]}{r}d\tau-\frac{1}{4\pi}grad\int\int\int\frac{[\rho]}{r}d\tau \\ +\frac{1}{4\pi c^2}\left[v,\ curl\int\int\int\frac{[\rho u]}{r}d\tau\right]. \tag{6}$$

In this form the electrodynamical problem has the appearance of being solved, for if we know the values of ρ and of ρu, we are in a position, through (6), to determine the force on a charge and hence its motion. It is not, however, easy, or indeed possible, to know under all circumstances what values to substitute in (6). This relation may, however, be regarded as a statement of our belief that it is always possible to assign values to ρ and ρu which satisfy the equation of continuity and which, when substituted in (6) will enable us correctly to calculate the motion of the charges.

B. *The basic problem of unipolar induction.*—The problem of unipolar induction is to account for the electric field in the vicinity of a rotating magnet. A magnet is a complicated structure made up of elementary magnets which are in some way connected with the atomic structure of the material of which the magnet is composed. Of the nature of the ultimate magnetic particle we know little or nothing. From the viewpoint of the theory sketched above it must be some form of current whirl of electricity, although the restriction there made that *div* $H=O$ may have been too drastic. It is entirely possible that true magnetic doublets exist. In any event, however, the unipolar induction problem, in the final analysis, reduces to the determination of the electric field in the vicinity of a moving elementary magnet.[1]

To solve the problem in the manner indicated in equation (6), above, requires that we know the values of ρ and of ρu for the moving magnet, assumed to be some sort of current whirl. But unfortunately we do not know their values even for the resting whirl. True we may postulate values which, when substituted in (5) will make E zero and give H the value appropriate to an elementary magnet, but that is not all that is necessary. It is essential that relation (6) be satisfied by the motion of the component parts of the whirl. These remarks may appear trivial since, after all, we do not care how the whirl is constructed so long as we can calculate its external effects. A little thought will show, how-

[1] Swann, Phys. Rev., 15, 365, 1920, makes this problem the basis of his treatment of the theory of unipolar induction.

ever, that to calculate the field of the whirl after it has been set in motion, in terms of what it was before, necessarily involves a knowledge of how its structure is modified by the forces brought into play by its motion. To proceed requires that we bridge this gap in our knowledge by the introduction of an assumption not explicitly contained in the above theory. The test of the assumption will be the correctness of the results it predicts.

Our immediate inclination is to assign to the moving whirl a density $\rho'=\rho=O$ and a value of $\rho'u'=\rho(u+v)$ where ρ and u refer to the values for the resting whirl. If we do this and calculate the force on a charge at rest in the vicinity of the moving whirl, we arrive at a result which we regard as absurd, namely, that the charge will experience a force having a component in the direction of the motion of the whirl.

The safest way to proceed to the result is to give up the semblance of predicting the field of the moving whirl, and instead to assume the field suggested by the theory of relativity. We may then make use of (6) to find the values of ρ and of ρu which would have to be used in order to give that result. This is essentially the method used by Swann.[1] It turned out that, due to its motion, the density of the electricity of the whirl so changes that it becomes the equivalent, electrostatically, of an electric doublet of moment

$$N=\frac{[vM]}{c}$$

where M is the magnetic moment of the whirl. This means that in the force equation for the resting charge,

$$F=-\frac{1}{c}\frac{\partial A}{\partial t}-grad\ \phi$$

the scalar potential, ϕ, is not zero but is the electrostatic potential due to an electrical doublet of the above moment. Under these circumstances the force, F, will be

$$-\frac{[vH]}{c}$$

as is suggested by the theory of relativity.

2. The above method of solving the problem, by assuming the result, is a little unsatisfactory in that it seems a confession of weakness on the part of electromagnetic theory. It may be of interest to inquire to what extent the theory itself has the power to suggest the solution.

Let us suppose that the elementary magnet is made up of any system of charges in a stationary state of motion. Let the force on the electricity at any point of the system be

$$F_0=E+\frac{[uH]}{c}$$

[1] Swann, loc. cit.

Let the system be set in uniform rectilinear motion with the velocity v, and assume that, at any rate for small values of v, the magnetic field and the force F_0 necessary to maintain stability of motion of the electricity, remain the same. Due to the motion of the system there will be an additional force, $\frac{[vH]}{c}$, acting on the electricity and hence to keep F_0 constant requires that a redistribution of electricity take place of such a nature that an electrostatic field $E' = -\frac{[vH]}{c}$ is created. If ρ_0 and ρ refer to the charge density in the system before and after being set in motion, we have:

$$\begin{aligned} \rho_0 &= div\ E \\ \rho &= div\left(E - \frac{[vH]}{c}\right) \\ &= \rho_0 - div\frac{[vH]}{c} \end{aligned}$$

and it is seen that the motion has produced an excess density of electricity

$$\Delta\rho = -div\frac{[vH]}{c} = \frac{1}{c}(v,\ curl\ H)$$

at each point of the current system.

We are now in a position to calculate that portion of the electric field in the vicinity of the moving system which is due to its motion. The scalar potential, ϕ, is given by:

$$\begin{aligned} \phi &= \frac{1}{4\pi}\int\int\int\frac{[\Delta\rho]}{r}d\tau = \frac{1}{4\pi c}\int\int\int\frac{[(v,\ curl\ H)]}{r}d\tau \\ &= \frac{1}{c}(v, A) \end{aligned}$$

where A is the vector potential of the system. Hence:

$$\begin{aligned} E &= -\frac{1}{c}\frac{\partial A}{\partial t} - grad\ \phi \\ &= -\frac{[vH]}{c} \end{aligned}$$

We therefore conclude that, in so far as the above assumptions are justified, any system of electric currents when in motion will, in virtue of a redistribution of its charges, exert a force on a resting charge in its vicinity given by:

$$E = -\frac{[vH]}{c}$$

where v is the velocity of the current system. This is a generalization of the statement made in the preceding section for the Amperian whirl. It is also applicable to a current system of finite size such as a solenoid.

3. *Field due to a true magnetic doublet in uniform motion.*—Admitting the possibility of the existence of true magnetic doublets, it is of interest to examine what the theory has to say about the effect their motion would have on an electric charge in their vicinity. To examine this point let us write the field equation in the symmetrical form:

$$\begin{aligned} \rho_e u_e + \dot{E} &= c \, curl \, H \\ \rho_m u_m + \dot{H} &= -c \, curl \, E \\ div \, E &= \rho_e \\ div \, H &= \rho_m \end{aligned} \tag{7}$$

Written in this way they enable us to take account of the possible existence of true magnetic doublets.

These equations may also be integrated in terms of potentials:

$$\begin{aligned} \phi_e &= \frac{1}{4\pi} \int\int\int \frac{[\rho_e]}{r} d\tau \\ A_e &= \frac{1}{4\pi c} \int\int\int \frac{[\rho_e u_e]}{r} d\tau \\ \phi_m &= \frac{1}{4\pi} \int\int\int \frac{[\rho_m]}{r} d\tau \\ A_m &= \frac{1}{4\pi c} \int\int\int \frac{[\rho_m u_m]}{r} d\tau \end{aligned} \tag{8}$$

which are subject to the restrictions,

$$\begin{aligned} div \, A_e &= -\frac{1}{c} \frac{\partial \phi_e}{\partial t} \\ div \, A_m &= -\frac{1}{c} \frac{\partial \phi_m}{\partial t} \end{aligned} \tag{9}$$

In terms of these potentials,

$$\begin{aligned} E &= -\frac{1}{c} \frac{\partial A_e}{\partial t} - grad \, \phi_e - curl \, A_m \\ H &= -\frac{1}{c} \frac{\partial A_m}{\partial t} - grad \, \phi_m + curl \, A_e \end{aligned} \tag{10}$$

The "force" equation has the same form as before except that here one must substitute the extended values for E and H given by (10). Thus:

$$F = -\frac{1}{c} \frac{\partial A_e}{\partial t} - grad \, \phi_e - curl \, A_m + \frac{[vH]}{c} \tag{11}$$

We may now examine the problem of finding the electric field in the vicinity of a true magnetic doublet in uniform motion. When the doublet is at rest we have:

$$\phi_e = 0$$
$$A_e = 0$$
$$\phi_m = M_x \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + M_y \frac{\partial}{\partial y}\left(\frac{1}{r}\right) + M_z \frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$
$$A_m = 0$$

When the doublet is set in motion with the uniform velocity, v, the obvious assumption to make is that

$$\phi'_e = \phi_e = O,\ A'_e = A_e = O,\ \phi'_m = \phi_m,\ A'_m = \frac{v}{c}\phi_m$$

Let the motion be along the axis of x with the uniform velocity v. Then clearly:

$$\frac{\partial A'_m}{\partial t} = -v\frac{\partial A'_m}{\partial x}$$

Substituting these values in (10) gives:

$$\begin{aligned}
E'_x &= 0 \\
E'_y &= -\frac{\partial A_m}{\partial z} = \frac{v}{c}H_z \\
E'_z &= \frac{\partial A_m}{\partial y} = -\frac{v}{c}H_y \\
H'_x &= -\frac{v^2}{c^2}\frac{\partial \phi_m}{\partial x} - \frac{\partial \phi_m}{\partial x} = \left(1+\frac{v^2}{c^2}\right)H_x \\
H'_y &= H_y \\
H'_z &= H_z
\end{aligned} \tag{12}$$

The electrical part of this field may readily be recognized as the field due to an electrostatic doublet, situated at the magnetic doublet with its axis perpendicular to the direction of motion and to the axis of the magnetic doublet, and of moment

$$N = \frac{[vM]}{c}$$

This is the same result as that obtained above, assuming that the elementary magnet was an Amperian whirl. The magnetic field of the doublet is unchanged by its motion except by terms of the order of $\frac{v^2}{c^2}$

It will be noted that on this view of things the term, $-curl\ A_m$, takes the part in the "force" equation (11) played by $-grad\ \phi$ on the other view. The scalar potential ϕ_e is here reserved for true electric charges.

As regards the question of the power of the above theory to predict the result, we may note that the solution depends upon the values which we assign to the quantities ρ'_e, $\rho'_e u'_e$, ρ'_m, and $\rho'_m u'_m$. The theory does not force any particular choice of these quantities upon us and in that sense has not the power to predict the result, at any rate in the present state of our knowledge.

4. *Rotation of a magnetic doublet about its own axis.*—We may now inquire what the theory has to say about the field in the vicinity of a magnetic doublet which is rotating about its own axis. In so far as it has the power to say anything it suggests that for the rotating doublet

$$\phi_e' = \phi_e = 0,\ A_e' = A_e = 0,\ \phi_m' = \phi_m,\ A_m' = A_m = 0$$

and thus that the field due to the rotating doublet is exactly the same as that due to the stationary one. If our elementary magnet is an Amperian whirl or any circular current system the same result is also strongly suggested as was pointed out by Pegram[1] for the case of a rotating solenoid and by Swann[2] for the Amperian whirl. Rotation of a circular current does not alter the differential velocity of the positive and negative electricity upon which alone the magnetic field depends.

It may be of interest in this connection to point out one way in which a current circuit is not equivalent, in respect of its external field, to a magnetic shell of uniform strength having the same boundary. For, as noted above, a circular current loop set in rotation about its axis would be surrounded by no electric field. A rotating magnetic shell, on the other hand, would be surrounded by an electric field since each elementary magnetic doublet of the shell would become, in addition, an electric doublet with axis radially out from the axis of rotation. There is thus in principle a fundamental difference between the unipolar induction effects of a rotating solenoid and those of a rotating material magnet.

As regards motion of translation, an electric current loop and a magnetic shell are equivalent in their effects.

To sum up the results of the last sections, it has been shown that, whatever the nature of the magnetic particle, or whatever the nature of the magnetic system, it will, when set in uniform translatory motion, be surrounded by an electric field which is, at any point, given by the expression

$$E = -\frac{[vH]}{c}$$

where v is the velocity of translation of the magnetic system. In this statement is contained the sole support which electromagnetic theory lends to the so-called moving line theory which regards the appearance of the electric field as due to the motion through space of lines of magnetic induction which are imagined rigidly attached to the magnetic system. It is to be especially noted that v refers solely to the translatory motion of the system and hence, in applying the moving line theory, the

[1] Pegram, Phys. Rev., 10, 591, 1917.
[2] Swann, *loc. cit.*

lines of magnetic induction must be regarded as partaking of the translatory motion of the system but not of the rotatory motion.

5. *The field surrounding a symmetrical magnet rotating about its axis of symmetry.*—In all unipolar induction experiments an essential feature is an axially symmetrical magnet rotating about its axis of symmetry. The theory sketched above will enable us to determine the field surrounding such a system. The electric field intensity, E, will be given at any point by:

$$E = -\frac{1}{c}\frac{\partial A_e}{\partial t} - grad\ \phi_e - curl\ A_m$$

Since the magnet is symmetrical and is rotating about its axis of symmetry the term $-\frac{1}{c}\frac{\partial A_e}{\partial t} = 0$. The term $-grad\ \phi_e$ is the ordinary electrostatic field due to the possible presence of electrical charges. The term $-curl\ A_m$ takes account of the field due to the motion of the magnetic doublets of which the magnet is composed. This latter field will be of a kind given by supposing that each magnetic doublet becomes, in addition, an electric doublet of moment

$$N = \frac{[vM]}{c} = \frac{\omega r M}{c}$$

where ω is the angular velocity of the magnet and r is the distance of the doublet from the axis of rotation. The net result for the whole magnet will be what we may term a fictitious polarization, P_0, of magnitude, at any point,

$$P_0 = \frac{\omega r I}{c} \quad (13)$$

The direction of this polarization will be in a plane through the axis of rotation and perpendicular to I.

A charge rotating with the magnet will experience an average force

$$F = -grad\ \phi_e - curl\ A_m + \frac{[vB]}{c}$$

where the last term results from its motion in the magnetic field whose true average value inside the magnet is B.

To determine what happens under this field it is necessary to know something about the nature of the material composing the magnet. If it be a dielectric the force acting on its charges will produce in it a true polarization of the ordinary type. Assuming the usual constitutive relation we may write this polarization as:

$$P = (\epsilon - 1)\ F = (\epsilon - 1)\left(-grad\ \phi_e - curl\ A_m + \frac{[vB]}{c}\right) \quad (14)$$

The field outside will be that due to the fictitious polarization, P_0, the true polarization, P, and whatever charges may be induced on conductors or dielectrics by these polarizations.

If the magnet be a conductor, electricity will move under the force, F, until, if possible, the force, F, is zero within the conductor. This will be possible provided F is derivable from a potential, and, given the symmetry of B, this is seen to be the case. Hence, if the magnet is a conductor, electricity will flow in it until at every point inside,

$$F=0=-grad\ \phi_e\ -curl\ A_m+\frac{[vB]}{c}$$

a relation which permits the calculation of ϕ_e.

The electric field strength, E, will be given by

$$E=-grad\ \phi_e-\ curl A_m,$$

It will be noted that that portion of $-grad\ \phi_e$ which cancelled $-curl\ A_m$ inside the magnet will do so outside as well, and hence:

$$E=-grad\ \phi'_e \tag{15}$$

where ϕ'_e is the electrostatic potential of the distribution of charge in the conducting magnet which is just necessary to cancel out, at points inside it, the motional intensity $\frac{[vB]}{c}$. Equation (15) contains the complete specification of the field due to a rotating symmetrical conducting magnet. Special cases may be solved explicitly. Swann[1] has worked out the case of a uniformly magnetized sphere and Rognley[2] that of a uniformly magnetized ellipsoid of revolution. It will be noted that in all cases the field is derivable from a potential and therefore will be completely screened off by any earthed shield surrounding the magnet but not partaking of its motion.

As a special application of equation (15) we may calculate the difference in potential between two points, 1 and 2, inside the magnet. We will have:

$$V_2-V_1=-\int_1^2 E_s ds=\int_1^2 \frac{[vB]_s}{c}\ ds$$

where the integral is to be taken over any line joining the two points. Since the field, E, is everywhere radial the two points and the path of integration joining them may, without loss of generality, be taken in the same meridian plane. Under these circumstances

$$[vB]_s=\omega r B_n$$

[1] Swann, *loc. cit.*
[2] Rognley, Phys. Rev., 19, 609, 1922.

where B_n is the component of B lying in the meridian plane and perpendicular to ds, the other component being parallel to ds. We may therefore write:

$$V_2 - V_1 = \frac{1}{c}\int_1^2 rB_n ds = \frac{1}{2\pi c}\int\int B_n dS \qquad (16)$$

where the surface integral is taken over the surface of revolution swept out by the path of integration in one revolution of the magnet. Clearly, since $div\ B = O$, we may write:

$$V_2 - V_1 = \frac{1}{2\pi c}\int\int B_n dS \qquad (17)$$

where the surface integral is taken over *any* surface having as boundaries the two circles swept out by the points 1 and 2 in one revolution of the magnet.

It may be noted that equation (17) also gives the potential difference between two points in any symmetrical conductor, not necessarily magnetic, rotating in a magnetic field which is symmetrical about the axis of rotation. If the magnetic field is not symmetrical the field will not be derivable from a potential and eddy currents will be set up in the conductor.

III

Unipolar Induction Experiments

A. *Experiments with closed circuits.*—The earlier experiments[1] differed but little in principle from Faraday's original experiment. The essential features in all of them were: (1) a symmetrical magnet, M, capable of being rotated about its axis of symmetry, and (2) a conducting circuit composed of two parts—one of which, A, rotated about the axis of the magnet, while the other, B, remained fixed. Brushes or mercury troughs maintained contact between A and B. Frequently the magnet itself formed the rotating part, A, of the circuit.

The experiments all showed that:—(1) an electromotive force is generated in the circuit by the rotation of A in the magnetic field, and this is true whether A is the magnet itself or not; (2) if A is separate from the magnet the electromotive force is entirely independent of

[1] For example, W. Weber, Pogg. Ann., 42, 1841; Pluecker, Pogg. Ann., 87, 351, 1852; Edlund, Ann. d. chim. et d. phys. (5) 16, 49, 1879; Exner and Czermack, Sits. ber. d. k. Akad. d. Wissensch. zu Wien 94, 357, 1886; Budde, Wied. Ann., 30, 358, 1887; Hoppe, Wied. Ann., 28, 483, 1886; Hoppe, Wied. Ann., 29, 544, 1886; Hoppe, Wied. Ann., 32, 297, 1887; Lecher, Wien. Ber., 53, 1894; Wied. Ann., 54, 276, 1895; Wied Ann., 69, 781, 1899.

whether the magnet is rotating or not. The integrated value of the electromotive force around the circuit was found in all cases to be:

$$\frac{\omega}{2\pi c}\int\int B_n\, dS$$

where ω is the angular velocity of rotation of A and where the integral represents the flux of magnetic induction through any cylindrical surface having as boundaries the circles swept out around the axis of revolution by the two points of A which are in contact with B. As may be noted by comparison with equation (17) above, this result is predicted by the theory there developed.

The usual basis employed by the early investigators for the calculation of the electromotive force in these experiments was derived from the Faraday-Neumann law according to which the electromotive force around a closed circuit is proportional to the rate of diminution of the flux of magnetic induction through the circuit. As is well known, this integral law will be predicted by a theory which states that the electromotive force at any portion of the circuit is proportional to the rate at which that portion of the circuit is cutting across the lines of magnetic induction.

From the point of view of this theory, the lines of magnetic induction acquired at least enough of reality that one could speak of cutting across them and of their being carried along by the magnet. It is only natural, therefore, that in the unipolar induction experiment the question should have been raised as to whether, when the magnet rotates, they are carried around as though rigidly attached to it, or whether they remain stationary while the magnet rotates through them; in other words, whether the seat of the electromotive force is in the stationary or in the moving part of the circuit. Indeed, this question soon became practically the sole point of interest in the experiments, and many and varied were the modifications introduced in the hope of throwing light on this point. As Beer[1] and T. Preston[2] clearly pointed out, however, both hypotheses gave exactly the same result for the integrated value of the electromotive force around a closed conducting circuit and hence measurements depending on this integrated value were powerless to discriminate between them. For on the resting line hypothesis the rotating part of the circuit is cutting the lines of induction, and the electromotive force at any point is $\frac{[vB]}{c}$. Integrating this between any two points on the surface of the rotating conductor gives, (*cf.* equation (17)):

$$\int_1^2 F_s ds = \frac{\omega}{2\pi c}\int\int B_n\, dS$$

[1] Beer, Pogg. Ann., 87, 351, 1852.
[2] T. Preston, Phil. Mag., (5), 19, 131, 1885.

where the surface integral has the same significance as before. The resting part of the circuit will contribute nothing to the integral and hence this is the total electromotive force around the circuit. If, on the other hand, the lines of force rotate with the magnet both the moving and the fixed parts of the circuit will contribute to the integrated value of the electromotive force, except in the special case where the rotating part of the circuit is rotating with the same angular velocity as the magnet. But a simple calculation will show that the result for the whole circuit is exactly the same as on the other hypothesis.

Both hypotheses had strong adherents,—for the stationary lines, Faraday, Pluecker, Lecher; for the rotating lines, Weber, Preston, Hertz, Lodge, Rayleigh. Faraday[1] very clearly stated his opinion in these words:—"When lines of force are spoken of as crossing a circuit, it must be considered as effected by the translation of a magnet. No mere rotation of a bar magnet on its axis produces any inductive effects on circuits exterior to it. The system of power about the magnet must not be considered as rotating with the magnet any more than the rays of light which emanate from the sun are supposed to revolve with the sun. The magnet may even in certain cases, be considered as revolving amongst its own forces, and producing a full electric effect sensible at the galvanometer." It is interesting to note that with a little more precision of statement, in which one would speak of the fields of the individual magnetic doublets of which the magnet is composed rather than of the field of the magnet as a whole, the preceding might very well have been written at the present time. It expresses about as well as could have been expressed at that time the present view of the moving line theory.

Clearly, though, the arguments advanced from either view point could be little more than expressions of opinion until some way of experimentally differentiating the two hypotheses could be devised.

A number of investigators had from time to time expressed the belief that an electrostatic examination of the field surrounding the rotating magnet would discriminate between the two views. T. Preston[2] proposed such an experiment and Hertz agreed that the results would be capable of deciding the matter. Unfortunately the experimental facilities available at that time were wholly inadequate for the success of the measurements. Lecher[3] attempted some electrostatic experiments but without success.

About 1900 another view began to be in evidence. Poincaré[4] and Abraham[5] both dismissed the whole question of the rotation or non-

[1] Faraday, Phil. Trans., 1852, p. 31.
[2] T. Preston, Phil. Mag., (5), 31, 100, 1891.
[3] Lecher, *loc. cit.*
[4] Poincaré, L'Eclairage Electrique, 23, 41, 1900.
[5] Abraham, Theorie d. Elektrizitaet, Vol. 1, p. 418.

rotation of the lines of induction by saying it had no meaning. Poincaré attempted to demonstrate this by pointing out that for closed circuits both hypotheses agreed in giving correct results but that for open circuits both agreed in giving wrong results. He discusses the theory of unipolar induction from the energy standpoint, but the expression used in finding the electromotive force around a closed circuit is the analytical equivalent of assuming it to be:

$$\int \frac{[vB]}{c} ds \tag{18}$$

exactly as before. This expression, as we have seen above, is true in general for a conducting circuit only. Nevertheless, in discussing a proposed experiment on open circuits, an experiment essentially the same as those afterward carried out by Blondlot[1], Wilson[2], and Barnett[3], Poincaré uses the expression in calculating the difference in potential between two points on the surface of an ebonite ring rotating in a magnetic field. Use of expression (18) for dielectrics is the characteristic feature of Hertz' theory of moving media and leads to an incorrect result as shown later by the experiments above mentioned. In discussing the light which open circuit experiments would throw upon the question of whether or not the lines of induction rotate, he adopts, for purposes of criticism, an unusual moving line theory according to which dielectrics are completely inert to the motion through them of lines of induction. It is not to be wondered at that he dismissed such a theory as meaningless.

B. *Electrostatic experiments.*—On the experimental side the matter rested here until about 1912 when a series of experiments by Kennard[4] and Barnett[5] were reported. These were experiments in which the displacement of electricity on a conductor in the neighborhood of a rotating magnet or solenoid was looked for. The experimental arrangements in all of them were very similar. The essential features were: (1) a magnet or solenoid mounted to rotate about the axis of symmetry of its magnetic field, and (2) a coaxial cylindrical condenser sometimes also mounted to rotate. In all of them the rotating magnet was screened from the condenser by an earthed shield. The purpose of this was to shield the condenser from spurious effects and at the same time to eliminate any effect due to the electrostatic field which, on the resting line theory, would surround the rotating magnet.

[1] Blondlot, Jour. de Phys., Jan., 1902.
[2] Wilson, H. A., Roy. Soc. Trans., A, Sept., 1904.
[3] Barnett, Phys. Rev., 27, 425, 1908.
[4] Kennard, Phil. Mag., 23, 937, 1912.
Phys. Zt., 13, 1155, 1912, and 14, 250, 1913.
Phys. Rev., 1, 355, 1913.
Phil. Mag., 33, 179, 1917.
[5] Barnett, Phys. Zt., 13, 803, 1912; 14, 251, 1913.
Phys. Rev., 35, 323, 1912.

The usual experimental procedure was as follows: (1) set the magnet in rotation, (2) connect the two cylinders of the condenser momentarily by a wire, (3) break the connection, (4) stop the rotation of the magnet, and (5) test the insulated cylinder for charge. The results all agreed in showing no effect comparable with that which would have been obtained had the cylindrical condenser been rotated in the field of the magnet.

On the basis of the theory developed above, no effect should have been obtained since, according to it, the field surrounding a rotating magnet is due entirely to a distribution of charge throughout the rotating magnet and would be completely screened from the condenser by the earthed shield surrounding the magnet.

We may now consider the relation between the negative results of these experiments and the vexatious question of whether or not the lines of magnetic induction rotate with the magnet. Kennard and Barnett differed completely in their interpretation of the results. Kennard concluded that here was definite evidence that the lines of induction did not rotate with the magnet, whereas Barnett argued that the experiments had nothing to say on this point. It is always very difficult to be sure that one understands correctly the view point of others and especially is this true in the field of electromagnetic theory into which so many symbols and names of things are introduced and so few defined. In the writer's opinion, however, the difference between the conclusions of these investigators is due primarily to a difference in what they have agreed to mean by the rotating line theory. Kennard's view, and perhaps the usual one, is that the rotating line theory requires that the force on a resting charge in the vicinity of an uncharged rotating magnet be given by an expression

$$F = -\frac{[vB]}{c}$$

where v is the linear velocity, at the charge, of a line of induction which is imagined fixed rigidly to the magnet and rotating with it. On this view the results of the above experiments definitely decide against the theory.

Barnett argued, however, that the expression for the force must be modified to include a term, say E, which would express the force on the charge due to a "displacement" in the aether brought about by the motion of the lines of induction through it. This displacement in the aether is assumed to be of just the right amount to cancel completely the force on the charge; that is, $E = \frac{[vB]}{c}$. If one is at liberty to assume that this field, E, is due to a fictitious distribution of electricity in the aether, the density of that distribution would be $div\ E$. Regarding the aether as a dielectric, this would mean a polarization of magnitude such that $-div\ P = div\ E = div\ \frac{[vB]}{c}$. That is, the polarization in the aether would be proportional to the motional intensity;

$$P = -\frac{[vB]}{c}$$

Toward the field due to the moving lines of induction it would therefore appear that the aether is assumed to be a dielectric of infinite specific inductive capacity. On the other hand it is necessary to assume that as regards fields of the ordinary electrostatic type it is a dielectric of unit specific inductive capacity, otherwise it would be impossible to exert any force on a charge imbedded in it.

With this view of matters one must be a little careful in considering the problem of the force on a charge due to the *rectilinear* motion of a magnetic doublet. Presumably the aether will react to the motion of the lines of induction which are carried along with the rectilinear motion of the doublet in the same way as it does to their rotation. The effect in this case would be also to annul all forces on the charge. To do this the aetherial polarization would be such as to deposit on the moving doublet a fictitious electric doublet of moment $-\frac{[vM]}{c}$. We know, however, that in this case there is a resultant force on the charge and it is therefore necessary to make some additional hypothesis. The most natural one to make is that, due to its motion, the magnetic doublet becomes in addition an electrical doublet of moment $\frac{[vM]}{c}$, in exactly the sense of the theory we have developed above. There can be no fundamental objection to this view of things but the expediency of introducing such an hypothetical aetherial polarization might be questioned. It would appear that unless there is some justification for it elsewhere in electromagnetic theory it would be purposeless to retain it here for the sake of a moving line theory.

Since he had no hope of discriminating between the two hypotheses as to the motion of the lines of induction, Barnett's chief interest in the experiments lay in the fact that in them the effect did not depend on the relative motion of the material parts of the system. As he points out, when the magnet is at rest and the condenser rotating an effect is obtained, whereas if the condenser is at rest and the magnet rotating none is obtained. There would seem to be no reason, a priori, for expecting the results to be the same, however, since the relative motions, in the general sense of the word, are by no means the same.

Recently Barnett[1] described an experiment in which he hoped to

[1] Barnett, Phys. Rev., 12, 95, 1918.

obtain evidence as to the existence or non-existence of the aether. Presumably the term aether is here applied to a medium of some such properties as those discussed above. In this experiment an electromagnet was mounted on a long pendulum support in such a way that its pole pieces, one on either side, could be swung past an earthed metal box containing a plane plate condenser with plates parallel to the line of motion of the magnet and to the magnetic field. The size of the pole pieces relative to the condenser was such that, at the moment when the magnet was in the center of its swing, the condenser and box might, without serious error, be regarded as at rest in the uniform field between two magnetic poles moving with a uniform velocity. The two poles of a similar but stationary electromagnet were brought into such a position that at the same moment the plane of demarkation between the lines of force from the two magnets, with poles in opposition, was coincident with the upper surface of the lower condenser plate. In this way Barnett hoped to screen the portion of the box below the lower condenser plate from the inductive effects produced by the moving magnet. Barnett[1] later expressed doubt as to the validity of this assumption; and its weakness is apparent. Under any circumstances, however, it is clear, as pointed out by Swann[2], that the earthed metal box surrounding the condenser would completely shield it from any electric field due to the moving magnet. The experimental results were entirely negative; no charging up of the condenser was observed. Barnett concluded that this negative result pointed to the existence of the aether and that it was inconsistent with the theory of relativity. That this conclusion was unjustified is evidenced by the fact that a negative result is predicted by a theory perfectly consistent with the theory of relativity.

In 1917 Pegram[3] completed a series of careful experiments essentially similar to those of Kennard and Barnett, in which a cylindrical condenser inside a rotating coaxial solenoid could be maintained at rest or rotated with the solenoid. With both solenoid and condenser rotating he obtained the charging up to be expected. With the solenoid rotating and the condenser at rest no effect was obtained, completely confirming the previous experiments.

In discussing the theory of unipolar induction Pegram emphasises the importance of the force equation in the form:

$$F = -\frac{1}{c}\frac{\partial A}{\partial t} - grad\ \phi + \frac{[vB]}{c}$$

[1] Barnett, Report on Electromagnetic Induction, A.I.E.E., Oct. 10, 1919. Phys. Rev., 15, 527, 1920.

[2] Swann, *loc. cit.*

[3] Pegram, Phys. Rev., 10, 591, 1917.

and in particular the importance of the first term $-\frac{1}{c}\frac{\partial A}{\partial t}$, involving the time rate of change of the vector potential. On the view of the theory given in II B 1 and 2, above, it would appear that in unipolar induction experiments it is rather the term $-grad\ \phi$ which is of greater importance, indeed in many cases the term $-\frac{1}{c}\frac{\partial A}{\partial t}$ is zero. For example, in one of the instances cited by Pegram, a charge finds itself inside two coaxial solenoids in which the currents are so adjusted that everywhere inside the inner solenoid the magnetic field is zero. The inner solenoid is then displaced transversely and the charge experiences a force. It would appear that in this case the term $-\frac{1}{c}\frac{\partial A}{\partial t}$ is zero and that the sole contribution to F is the term $-grad\ \phi$, which arises from the redistribution of charges in the wires of the solenoid discussed in II B 2 above.

Pegram clearly points out that on the crudest view of the electron theory of conduction it would be improbable that a solenoid rotating about its axis could exert a force on an electric charge in its vicinity and refers to Larmor[1] and Howe[2] who had also come to that conclusion. It is for this reason that there is an essential difference, not always realized, between unipolar induction experiments done with rotating solenoids and those done with rotating material magnets.

Recently Swann[3], who had previously[4] given an extremely lucid analysis of the whole theory of unipolar induction from the point of view of electrodynamical theory, reported on an experiment in which an effect due to the charging up of a sphere of iron rotating in a uniform magnetic field was measured. The axis of the rotating sphere was earthed and the charge which was induced on a metal shield completely surrounding the rotating sphere was measured. Considering the difficulties associated with measurements of that kind the results were remarkably consistent and were in complete agreement with the theory as outlined above.

C. *Status of the moving line theory.*—In all the experiments discussed thus far the experimental arrangement has been such that at most all that could be tested was whether the effects could be attributed to the usual view of the rotating line theory. They have decided against that view. It seems to have been usually assumed that there were but two possibilities, either the lines of induction rotate or they stand still. As first pointed out by Swann, however, and as is evident from

[1] Larmor, Roy. Soc. Phil. Trans. A, 727, 1895.
[2] G. Howe, Electrician, 76, 169, 1915.
[3] Swann, Phys. Rev., 19, 38, 1922.
[4] Swann, *loc. cit.*, ante.

II B 4 above, neither one of these views is in accord with electromagnetic theory. In order to be consistent with this theory it is necessary, if one wishes to think in terms of moving lines of induction, to imagine the lines of induction from each elementary magnetic doublet or Amperian whirl of which the magnet as a whole is composed, as carried along in the translatory motion of the doublet but not in any way partaking of its rotatory motion. Swann goes a step further and shows that if an elementary magnetic doublet is rotating about any axis making an angle with its own, that then one must regard the lines of force emanating from the two fictitious poles of the doublet as partaking of the translatory part of their motion only and not of their rotatory part.

It is interesting to inquire in how far this view of the motion of the lines of induction is verified by experiment. As pointed out above, the experiments of Kennard, Barnett, and Pegram indicated that the lines of induction did not partake of the rotation of the magnet as a whole. Is there any experiment in which, to account for the results, one must take account of the motion of translation of the elementary magnets composing the magnet? Clearly such an experiment would have to be one in which the rotating magnet is a non-conductor for, as pointed out above, if the magnet is a conductor the field due to the fictitious polarization produced by the individual translatory motions of the elementary magnets is completely cancelled out both at points inside and at points outside the magnet. In the case of the conducting magnet, therefore, the hypothesis of the stationary lines of induction will yield correct results. But this would no longer be the case if the magnet were a non-conductor.

The importance of doing an experiment of this kind on a non-conducting magnet was first pointed out by Einstein and Laub[1]. These authors showed from relativity considerations that if in the Wilson experiment, in which a cylindrical dielectric is rotated in a magnetic field parallel to its axis, the dielectric were at the same time magnetic the effect observed (for instance, the potential difference between the inner and outer coating of the dielectric) should involve the factor $\frac{\epsilon\mu-1}{\epsilon}$ instead of the factor $\frac{\mu(\epsilon-1)}{\epsilon}$ which they ascribed to the theory of Lorentz.

We may show, however, that a proper application of the theory of Lorentz to an experiment of this kind will not lead to an effect proportional to $\frac{\mu(\epsilon-1)}{\epsilon}$ but to $\frac{\epsilon\mu-1}{\epsilon}$ as suggested by relativity considera-

[1] Einstein and Laub, Ann. d. Phys., 26, 532, 1908.

tions. Let us consider for simplicity a magnetic dielectric in the form of a hollow cylinder rotating with uniform angular velocity, ω, about its axis in a uniform magnetic field parallel to the axis of rotation. Let I be the intensity of magnetization and suppose, as is customary, that

$$I=(\mu-1)H$$

where μ is the permeability of the dielectric. When the dielectric is set in rotation a fictitious polarization

$$P_0=\frac{\omega rI}{c}$$

will appear in conformity with the ideas developed above. This polarization is directed radially out from the axis of rotation and the electrostatic field due to it is clearly $-P_0$. This field, together with the motional intensity $\frac{\omega rB}{c}$, acting on a charge rotating with the dielectric will conspire to produce in the dielectric a true polarization, P. Since both the field $-P_0$ and $\frac{\omega rB}{c}$ are radial the resulting polarization will be radial and the electrostatic field due to it will likewise be $-P$. In equilibrium we shall have, assuming that there are no other charges anywhere,

$$\begin{aligned} P&=(\epsilon-1)\left(E+\frac{\omega rB}{c}\right)\\ &=(\epsilon-1)\left(-P-P_0+\frac{\omega rB}{c}\right)\end{aligned}$$

or

$$P=\frac{\epsilon-1}{\epsilon}\left(\frac{\omega rB}{c}-\frac{\omega rI}{c}\right)=\frac{\epsilon-1}{\epsilon}\left(\frac{\omega rH}{c}\right)$$

and substituting we find:

$$\begin{aligned} E&=-P-P_0=-\frac{\epsilon-1}{\epsilon}\,\frac{\omega rH}{c}-\frac{\omega rI}{c}\\ &=-\frac{\epsilon\mu-1}{\epsilon}\,\frac{\omega rH}{c}\end{aligned}$$

Failure to get this expression for the field is due to the neglect of the ficitious polarization in the dielectric produced by the motion of the magnetic doublets. The factor $\frac{\mu(\epsilon-1)}{\epsilon}$ would be predicted by a theory which regards the lines of induction of the magnet as stationary. To obtain the correct result on a moving line theory requires that we take account of the translatory motion of the lines of induction attached to the individual elementary doublets.

An experiment of the kind here discussed has been performed by M. Wilson and H. A. Wilson[1] using an artificially prepared magnetic dielectric composed of steel balls imbedded in paraffin. Their results were decisively in favor of the relation given by Einstein and Laub and are in complete agreement with the present theory. It might perhaps be argued that the results of this experiment are obvious since, after all, the dielectric is made up of many small magnets in rotation about a common axis, but then it might with equal right be argued that the fallacy of the stationary line theory is equally obvious.

IV

Summary

The electric field in the neighborhood of any symmetrical magnetic system spinning about its axis of symmetry is accounted for by the Maxwell-Lorentz theory to an extent determined by the correctness of certain assumptions which it is necessary to make. The field so calculated is in complete accord with all known experimental facts.

The field may, in all cases, be correctly calculated by a moving line theory in which the lines of induction of each elementary magnet of which the system is composed partake solely of the translatory part of the motion of that element.

A theory which postulates that the lines of magnetic induction rotate with the magnetic system gives incorrect results in general, but may be used to calculate the integrated value of the electromotive force around a closed conducting circuit, part of which is rotating, and part stationary.

A theory which postulates that the lines of induction stand still while the magnetic system rotates through them will yield correct results if the magnetic system is a conductor but incorrect results, in general, if it is a dielectric.

[1] M. Wilson and H. A. Wilson, Proc. Roy. Soc. London, A, 89, 99, 1913.

PART III

EQUATIONS FOR THE DESCRIPTION OF ELECTROMAGNETIC PHENOMENA

By H. BATEMAN

I. INTRODUCTORY REMARKS ON ELECTRICITY, SPACE AND TIME

The idea of motion implies the existence of some means of recognizing again and again the entity that moves. By extending the idea of a mathematical point we have the concept of a moving point which we shall call an *electrical point* and we may start with the fundamental hypothesis that three independent quantities (α, β, γ) are sufficient to specify an electrical point and distinguish it from others. To describe its motion analytically we need a system of space and time co-ordinates. Without entering into a discussion as to the nature of these co-ordinates or the manner in which they are obtained, we may remark that the motion of the electrical points is supposed to be described in any case by equations of the form

$$x=f_1(\alpha, \beta, \gamma, t), \quad y=f_2(\alpha, \beta, \gamma, t), \quad z=f_3(\alpha, \beta, \gamma, t) \tag{1}$$

where f_1, f_2, f_3, are single-valued functions of their arguments for a definite group of electrical points. By eliminating the parameters α, β and γ we may obtain a set of differential equations

$$\frac{dx}{v_x}=\frac{dy}{v_y}=\frac{dz}{v_z}=dt \tag{2}$$

for the paths of the electrical points of a group and if the equations (1) can be solved for α, β and γ we can express the solution of these equations in the form

$$\begin{aligned} \alpha(x, y, z, t) &= \text{constant}, \\ \beta(x, y, z, t) &= \text{constant}, \\ \gamma(x, y, z, t) &= \text{constant}. \end{aligned} \tag{3}$$

When the functions α, β and γ possess continuous derivatives we may infer from (2) and (3) that

$$\rho v_x=-\frac{\partial(\alpha,\beta,\gamma)}{\partial(y,z,t)}, \quad \rho v_y=-\frac{\partial(\alpha,\beta,\gamma)}{\partial(z,x,t)}, \quad \rho v_z=-\frac{\partial(\alpha,\beta,\gamma)}{\partial(x,y,t)}, \quad \rho=\frac{\partial(\alpha,\beta,\gamma)}{\partial(x,y,z)} \tag{4}$$

where ρ is some function of x, y, z and t, which we shall call the *density* of *electricity*. This function, ρ, satisfies the *equation of continuity*

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}\left(\rho v_x\right)+\frac{\partial}{\partial y}\left(\rho v_y\right)+\frac{\partial}{\partial z}\left(\rho v_z\right)=0 \tag{5}$$

and so we shall regard electricity as a substance.

Our assumption that α, β and γ are independent implies that the quantities ρv_x, ρv_y, ρv_z and ρ are not all zero. There may also be some moving points which cannot be distinguished from each other. This case would be included in the present analysis if γ were a function of α and β. The quantities ρv_x, ρv_y, and ρv_z and ρ would then be zero and a group of such moving points could not be regarded as representing electricity; moreover, the moving points having the same values of α, β and γ would generally lie on a moving curve. We might call such points *aethereal points* and introduce two different fundamental concepts, viz., *aether* and electricity, but it seems worth while to try to avoid the introduction of an aether at the outset, and to work with only one substance, viz., electricity.

So far nothing has been said about the geometry of our groups of moving points. Are we to assume that space is Euclidean or non-Euclidean? The view is widely held that space is neither physical nor metaphysical but conventional[1] and this is the view we shall adopt in the following discussion. The conventional space, then, is Euclidean for any individual observer, and the co-ordinates used in his analytical geometry have the properties of ideal quantities which may be imagined to have been obtained by ideal measurements in an ideal medium in which the electrical points move, under normal circumstances, along straight lines with constant velocity c. An event occurs when the direction of motion of an electrical point is changed. This may be the result of a collision with some other electrical point or it may be due to some other cause at present unknown. An electrical point which is continually colliding with others may move along a zig-zag path and in the limiting case when the collisions in a finite time are infinite in number the electrical point may appear to move in a curve or straight line with a velocity less than c, or it may appear to be stationary.

We may express the matter in another way by saying that it is on account of collisions between aethereal points that electrical points exist. Let us suppose that we have two sets of ∞' aethereal points specified by the equations

$$\alpha(x, y, z, t) = \alpha_0, \qquad \beta(x, y, z, t) = \beta_0 \tag{6}$$

and

$$\beta(x, y, z, t) = \beta_0, \qquad \gamma(x, y, z, t) = \gamma_0 \tag{7}$$

respectively. Calling the points of the first set P_1, P_2, P_3, . . . and those of the second set Q_1, Q_2, Q_3, . . . we may suppose that first P collides with Q_1 at R_1; that then P_2 collides with Q_2 at R_2 and so on. Even though all the P's may be moving along straight lines with velocity c and all the Q's likewise, yet the moving point R with parameters

α_0, β_0 and γ_0 may move along a curve with a velocity less than c. This point R possesses characteristics of both the P's and the Q's and its path may be regarded as the limit of a zig-zag path which consists of first, a small portion of P_1's rectilinear path, then a portion of Q_2's rectilinear path, then a portion of P_2's path and so on. (Fig. 1.)

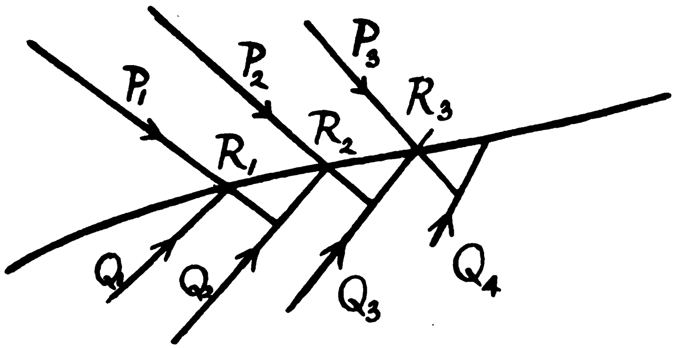

FIG. 1

We have tacitly assumed here that we really have some means of distinguishing between P_1, P_2, etc., so that these points can be regarded as electrical instead of aethereal. The parameters for P_1 may be $(\alpha_0, \beta_0, \bar{\gamma}_1)$, those for P_2 $(\alpha_0, \beta_0, \bar{\gamma}_2)$ and so on, where $\bar{\gamma}(x, y, z, t)$ is some function which is not the same as γ. In this sense the electrical point R which moves along the path R_1, R_2 . . . does not always consist of the same *fundamental electric point*, using this term to denote a point that moves with velocity c along a rectilinear path. The electrical point is in fact first P_1, then Q_2, then P_2 and so on.

The fundamental electrical points and our ideal system of space and time co-ordinates are auxiliary quantities which may be compared with parameters introduced in geometry to serve as the arguments of uniform functions occurring in the representation of quantities connected by one or more relations. Like these parameters they are probably not unique and may not have a simple physical meaning.*

We must distinguish, of course, between the ideal measurements just mentioned and actual measurements which are made with material bodies. The ideal medium and the ideal system of space-time measurements are adopted simply as the basis of a simple mathematical language and our hypothesis is that with a suitable choice of the constant, c, this language is quite adequate for the description of any conceivable phenomena. In other words, if we start with simple functions of our auxiliary parameters, we assume that we can, by suitable operations, build

* There may not, for instance, be a correspondence between the fundamental electrical points used by one observer and those used by another.

up more complex functions capable of representing the structure and behaviour of physical entities of an arbitrary nature. By identifying some of these complex functions with quantities that are actually measured we may eventually arrive at a theory of measurement.

An exact theory of measurement is difficult and so we shall first proceed on the assumption that our system of co-ordinates (x, y, z, t) is actually identical with one obtained by physical measurements and shall endeavour to formulate a scheme of equations which will account approximately for physical phenomena.

We shall assume now that x, y and z are rectangular co-ordinates and that the constant c is the velocity of light $(3\times 10^{10}\,\frac{cm.}{sec})$. We shall also regard (v_x, v_y, v_z) as the components of a velocity v which is a vector function of x, y, z, t for regions of space occupied by electricity. We shall use the vector notation of Gibbs in some of our equations and shall employ a notation that has become popular for the electric and magnetic quantities.

II. Theory of the Lorentz-Fitzgerald Contraction

In attempting to formulate the equations of motion of electricity we shall be guided by the hypothesis that bodies contract when in motion in such a way that an axis in the direction of motion is $\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$ times the length of a corresponding axis of a body at rest, while axes at right angles to the direction of motion are unaltered in length.[2] This is only a rough statement of the contraction hypothesis; a more precise statement will be given later.

We shall use the symbol ρ to denote the density of the *true* electricity and v its velocity—the term true electricity being used to denote the substance made up of electrical points that travel with velocities less than c and that arise from collisions between fundamental electric points which travel along straight lines with velocity c.

We now introduce the retarded potentials

$$\Phi=\frac{1}{4\pi}\int\frac{dV}{r}[\rho]_{t-\frac{r}{c}},\quad A=\frac{1}{4\pi c}\int\frac{dV}{r}[\rho v]_{t-\frac{r}{c}},\quad \Psi=\frac{1}{4\pi}\int\frac{dV}{r}\left[\rho\sqrt{1-\frac{v^2}{c^2}}\right]_{t-\frac{r}{c}} \tag{8}$$

$$dV=d\xi d\eta d\zeta,\quad r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2,\quad [\rho]_{t-\frac{r}{c}}=\rho\left(\xi,\eta,\zeta,t-\frac{r}{c}\right), \tag{9}$$

the field vectors

$$H=curl\,A,\qquad E=-\frac{1}{c}\frac{\partial A}{\partial t}-\nabla\Phi \tag{10}$$

and the tensor components

$$
\begin{aligned}
W&=\frac{1}{2}(E^2+H^2)+\frac{2}{c^2}\Psi\frac{\partial^2\Psi}{\partial t^2}+\left(\frac{\partial\Psi}{\partial x}\right)^2+\left(\frac{\partial\Psi}{\partial y}\right)^2+\left(\frac{\partial\Psi}{\partial z}\right)^2-\frac{1}{c^2}\left(\frac{\partial\Psi}{\partial t}\right)^2+C\\
S_x&=c[E_yH_z-E_zH_y]-2\Psi\frac{\partial^2\Psi}{\partial x\partial t}=c^2G_x\\
S_y&=c[E_zH_x-E_xH_z]-2\Psi\frac{\partial^2\Psi}{\partial y\partial t}=c^2G_y\\
S_z&=c[E_xH_y-E_yH_x]-2\Psi\frac{\partial^2\Psi}{\partial z\partial t}=c^2G_z\\
X_x&=\frac{1}{2}[E_x^2-E_y^2-E_z^2+H_x^2-H_y^2-H_z^2]-2\Psi\frac{\partial^2\Psi}{\partial x^2}+\Lambda\\
Y_y&=\frac{1}{2}[E_y^2-E_z^2-E_x^2+H_y^2-H_z^2-H_x^2]-2\Psi\frac{\partial^2\Psi}{\partial y^2}+\Lambda\\
Z_z&=\frac{1}{2}[E_z^2-E_x^2-E_y^2+H_z^2-H_x^2-H_y^2]-2\Psi\frac{\partial^2\Psi}{\partial z^2}+\Lambda\\
Y_z&=Z_y=E_yE_z+H_yH_z-2\Psi\frac{\partial^2\Psi}{\partial y\partial z}\\
Z_x&=X_z=E_zE_x+H_zH_x-2\Psi\frac{\partial^2\Psi}{\partial z\partial x}\\
X_y&=Y_x=E_xE_y+H_xH_y-2\Psi\frac{\partial^2\Psi}{\partial x\partial y}\\
\Lambda&=\left(\frac{\partial\Psi}{\partial x}\right)^2+\left(\frac{\partial\Psi}{\partial y}\right)^2+\left(\frac{\partial\Psi}{\partial z}\right)^2-\frac{1}{c^2}\left(\frac{\partial\Psi}{\partial t}\right)^2+C
\end{aligned}
\tag{11}
$$

The quantity W will be regarded as the density of energy, the vector S is supposed to specify the flux of energy and the vector G the momentum. The other quantities represent the six components of stress. C is a positive constant which is introduced so that W may never become negative. C has different values inside and outside electrons.

The equations of motion will be assumed to be

$$
\begin{aligned}
\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}-\frac{1}{c^2}\frac{\partial S_x}{\partial t}&=0\\
\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}-\frac{1}{c^2}\frac{\partial S_y}{\partial t}&=0\\
\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}-\frac{1}{c^2}\frac{\partial S_z}{\partial t}&=0\\
\frac{\partial S_x}{\partial x}+\frac{\partial S_y}{\partial y}+\frac{\partial S_z}{\partial z}+\frac{\partial W}{\partial t}&=0
\end{aligned}
\tag{12}
$$

These may be written in the form

$$\begin{aligned} \rho\left[E_x+\frac{v_y}{c}H_z-\frac{v_z}{c}H_y\right]+2\Psi\frac{\partial}{\partial x}\left[\rho\sqrt{1-\frac{v^2}{c^2}}\right]&=0 \\ -\rho(v\cdot E)+2\Psi\frac{\partial}{\partial t}\left[\rho\sqrt{1-\frac{v^2}{c^2}}\right]&=0 \end{aligned} \tag{13}$$

and, since $\rho\left[E+\frac{1}{c}(v\times H)\right]$ is usually regarded as *the electromagnetic force per unit volume*, $2\Psi\nabla\left[\rho\sqrt{-1\frac{v^2}{c^2}}\right]$ must be interpreted as *a non-electromagnetic force* which just balances the electromagnetic force.* The last equation may be regarded as *the principle of the conservation of energy*. The energy equation is thus adopted in its simplest form

$$\frac{\partial W}{\partial t}+div\ S=0$$

A more complicated form of the energy-equation has been discussed by A. Szarvassi† in relation to electromagnetic phenomena in moving media and he finds a general form of field equations with which it is consistent. His work belongs, however, to the subject of Section VIII. On account of its complexity, the energy-equation for a material medium has not been discussed in Section VIII.

If $\Psi\neq 0$ the equations (13) imply that

$$\frac{d}{dt}\left[\rho\sqrt{1-\frac{v^2}{c^2}}\right]=0 \tag{14}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z} \tag{15}$$

This equation tells us that $\rho\sqrt{1-\frac{v^2}{c^2}}$ remains constant during the motion of the electricity. This may be regarded as a precise statement of the contraction hypothesis. The condition can evidently be satisfied if an electron has a spherical boundary and concentric spherical layers of equal density when at rest, and becomes defoımed into an electron with an oblate spheroid as its boundary and concentric layers of equal density which are similar and similarly situated oblate spheroids, when set in motion, provided the axes of the spheroid are $\left(a\sqrt{1-\frac{v^2}{c^2}},\ a,\ a\right)$ when the velocity of the electron is v.

* A previous attempt (Phys. Rev., vol. 12, 1918, p. 477) to balance the electromagnetic force by another force and reduce the equations of motion to the form (12) is unsatisfactory because the equations $Y_z = Z_y$, $Z_x = X_z$, $X_y = Y_x$ are not satisfied

† Phys. Zeitschr. Bd. 10 (1909), p. 811, Wien Ber. Ia. Bd. 119 (1910), p. 281, Bd. 120 (1911), p. 337.

When a body made up of a number of discrete electric charges is set in motion we may infer that if the body remains rigid when in uniform motion all these charges will contract in the same ratio. To conclude that the body itself contracts in the same ratio we may make use of the transformations of the theory of relativity to prove that this condition is consistent with the equations of equilibrium expressed by equations (12). It is clear that we shall also be led to the equation (14) if we multiply all the terms depending on Ψ in the tensor components by the same constant factor k. The particular choice ($k=1$) of this factor has been made so that the total radiation per period from an electric pole describing a periodic orbit may be exactly zero. We assume that this is also true, with the same value of k, for the case of an electron of finite size but we do not feel sure that this assumption is justifiable. The choice $k=\frac{1}{2}$ would lead to much simpler equations for the determination of ρ in electrostatic problems but does not lead to non-radiating orbits for the case of an electric pole.

III. The Structure of Matter

Many attempts have been made to determine the form of the electron and many different hypotheses have been advanced; thus, we have the rigid electron of Abraham,[3] the deformable electron of Lorentz,[4] the ring electron of Parson[5] and Compton[6] and the electron recently invented by Page[7] in which there is a magnetic field instead of an electric field inside the electron. Attempts to account for the permanence of the electron have been made by Poincaré[8] and Page, but we shall not discuss them here; our object is to show that with a suitable law of density the electrostatic force $-\rho\nabla\Phi$ is just balanced by the new force, depending on the gradient of the density, which, in the electrostatic case, is represented by $2\Phi\nabla\rho$. It is clear that the forces balance if $\Phi=b\rho^2$ where b is a constant.[9] Assuming that the electron is bounded by a sphere of radius a and that the density, ρ, depends only on the distance r from the centre of this sphere, we assume that

$$\rho=A_0+A_2r^2+A_4r^4+\cdots \tag{16}$$

The corresponding potential is

$$\Phi=\frac{A_0}{2\cdot 3}(3a^2-r^2)+\frac{A_2}{4\cdot 5}(5a^4-r^4)+\frac{A_4}{6\cdot 7}(7a^6-r^6)+\cdots \tag{17}$$

Putting $\Phi=b\rho^2$ and equating coefficients, we find that

$$\rho=A_0\left[1-\frac{r^2}{12bA_0}-\frac{r^4}{720b^2A_0^2}-\frac{r^6}{100800b^3A_0^3}-\frac{31r^8}{(720)(5040)b^4A_0^4}\right.$$
$$\left.-\frac{971r^{10}}{(720)^2(2310)b^5A_0^5}-\frac{53456.5\,r^{12}}{(720)^2(2940)(429)b^6A_0^6}-\cdots\right] \tag{18}$$

where

$$0=F\left(\frac{a^2}{bA_0}\right)=1-\frac{1}{2}\frac{a^2}{bA_0}+\frac{1}{48}\left(\frac{a^2}{bA_0}\right)^2+\frac{1}{4320}\left(\frac{a^2}{bA_0}\right)^3+\frac{1}{80640}\left(\frac{a^2}{bA_0}\right)^4$$
$$+\frac{31}{(7200)(5040)}\left(\frac{a^2}{bA_0}\right)^5+\frac{971}{(720)^2(27720)}\left(\frac{a^2}{bA_0}\right)^6 \qquad (19)$$
$$+\cdots$$
$$+\frac{53456.5}{(720)^2(2940)(6006)}\left(\frac{a^2}{bA_0}\right)^7+\cdots$$

The equation $F(x)=0$ appears to have two positive roots; the first of these is approximately 2.21 and corresponds to a law of density in which ρ diminishes slowly in absolute magnitude from a maximum value A_0 at the centre to a minimum value .8 A_0 at the boundary $r=a$. The second root, if it exists, is difficult to determine accurately by the present method† owing to the slow convergence of the series for values of x in the neighbourhood of 10. The minimum value of $|\rho|$ at the boundary is probably very much less than the maximum value of $|\rho|$ at the centre.

If the electron is supposed to be situated in an external field of constant potential Ψ_0, the equation $F\left(\frac{a^2}{bA_0}\right)=0$ must be replaced by

$$F\left(\frac{a^2}{bA_0}\right)=\frac{\Psi^0}{bA_0^2}$$

and it appears that we cannot have the same values of ρ_{max}, ρ_{min} and the total charge as before. Now, on the other hand, if the electron were supposed to move from a state of equilibrium with no external field into a state of equilibruim in a field with potential Ψ_0, the quantities ρ_{max}, ρ_{min}, and the total charge might be expected to be the same in both cases. An explanation of this paradox may, perhaps, be obtained by considering the equilibrium of two electrons under their mutual influence. If for simplicity we imagine one electron to be the image of the other in a plane $x=0$, it is clear that equilibrium is possible only if each electron is deformed so that there are forces of attraction, arising from the density gradients, sufficient to balance the forces of repulsion.

Since ρ is constant on the boundary of the spherical electron, it must also be constant on the boundary of the deformed electron and so the boundary is an equipotential surface. The shape of the boundary may be supposed to be determined by the potential equation when the size and mutual distance of the electrons are known. The quantities to be

† A rough estimate, based on the first nine terms of the series, gives $x=11$. It is thought that the smaller root may give the electron, and the larger root the proton. A more accurate value of the smaller root is 2.209012572.

determined are thus the size, distance apart, total charge and the constant b. On the other hand by comparing each electron with the spherical electron we obtain three equations by comparing the total charges and values of ρ_{max} and ρ_{min} but there is also a fourth equation

$$\Phi_{max}=b(\rho_{max})^2 \tag{20}$$

which corresponds to our previous equation $F(x)=0$. We may thus expect there to be only a finite number of solutions of the problem and so it is possible that positions of equilibrium for two electrons (or two positive charges) exist only when the linear dimensions and mutual distance are connected by certain relations.

An exact solution of the problem is much to be desired. It may be that an electron which has been in equilibrium when alone can only be in equilibrium in certain particular fields and that some readjustment of charge is necessary before it can be in equilibrium in an arbitrary field. It seems likely, too, that a state of steady motion of a system of discrete charges may be possible only when the size of each orbit is related in some way to the size of each charge. The present analysis may, then, lead eventually to Bohr's theory of the atom, but it is full of difficulties.

If b and A_0 are given constant negative values and a negative constant potential Ψ_0 is added to (17) it appears that equilibrium is possible only with a smaller value of a and a smaller total charge. This does not mean, however, that an electron would necessarily lose charge when brought into a field with a negative potential. The effect of the negative potential would be, in fact, to cause a shrinkage. This shrinkage could not simply continue until a new state of equilibrium, with a different value of b, was reached because $\rho\sqrt{1-v^2/c^2}$ remains constant If $|\rho|$ increases, v must increase also. Unfortunately we cannot follow these steps which occur when an electron passes from one field to another and so we cannot say at present whether the electron loses charge or not. If two electrons were made to approach one another we might expect both to shrink and be deformed; but it is necessary to take into account the nature of the agency which causes the approach. If it is simply speed of translation, we may expect the kinetic energy to be transformed into the potential energy associated with the increased masses of the two electrons.

A point of some interest is, that the binding forces which hold together a number of electric charges of the same sign can apparently be weakened by decreasing the absolute magnitude of the potential Ψ. It may be that a radioactive transformation occurs when a large number of nuclear (or external) electrons accidently concentrate in the immediate neighbourhood of one portion of the nucleus[10] and weaken the binding forces by lowering the positive potential. It is unlikely that radioactivity can be controlled* because a large negative potential is probably needed to bring the positive potential down to a critical value. It is hardly likely that a sufficiently high negative potential would be obtained when a number of electrons in a substance approached one another as a result of the heat motion.

At a high temperature when many electrons have been driven out of a substance, the positive potential in the neighbourhood of an atomic nucleus may be larger than usual and if the pressure is also very great we may have the right conditions for the formation of complex nuclei. The complex elements may thus be formed in the stars.

The present analysis thus raises many interesting questions that need answering and may eventually indicate the reason why all electrons carry approximately the same charge. It also suggests the possibility that in a chemical compound we may have two electrons close together and in equilibrium under their mutual attractions and repulsions, both electrons being deformed. Such a pair of electrons might act as a chemical bond between two atoms. According to Lewis[11] and Langmuir[12] the pair of electrons is one of the most stable groups.

It should be mentioned that experiments on the deflection of α-particles by atomic nuclei give the usual inverse square law of repulsion, because the α-particles travel at such a high speed that there is not enough time for the α-particle and atomic nucleus to become deformed when they are close together and give rise to mutual attractions sufficient to appreciably modify the repulsion. If an atomic nucleus and an α-particle could be made to approach each other slowly there might be some chance of a combination.

IV. The Field of a Moving Electric Pole

Exact expressions for the electromagnetic potentials belonging to the field of a moving electric pole were deduced from the corresponding potentials for a continuous distribution of electricity by A. Liénard[13] and E. Wiechert,[14] approximate forms having been used previously by J. Larmor.[15] The exact potentials are

$$A=e\frac{v}{M}, \quad \Phi=e\frac{c}{M} \tag{21}$$

where v is the velocity of the pole at time τ, $[\xi(\tau), \eta(\tau)\zeta(\tau)]$ are the rectangular co-ordinates of the pole at this instant, $[\xi'(\tau), \eta'(\tau), \zeta'(\tau)]$ its com-

* We are thinking here of control by means of agents that do not owe their existence to radioactivity. Rutherford has succeeded in breaking up some atoms with the aid of α-particles.

ponent velocities, $-4\pi e$ its electric charge, and M a quantity defined by the equations

$$M=[x-\xi(\tau)]\xi'(\tau)+[y-\eta(\tau)]\eta'(\tau)+[z-\zeta(\tau)]\zeta'(\tau)-c^2(t-\tau), \\ =r(v_r-c), \tag{22}$$

$$r^2=[x-\xi(\tau)]^2+[y-\eta(\tau)]^2+[z-\zeta(\tau)]^2=c^2(t-\tau)^2. \tag{23}$$

The last equation, combined with the inequality $\tau<t$, associates a time τ with each space-time point (x, y, z, t). If the velocity v is always less than c the time τ associated with (x, y, z, t) is unique and increases as t increases if (x, y, z) remains stationary or moves with a velocity less than c. The point $[\xi(\tau), \eta(\tau), \zeta(\tau)]$ is called the *effective position*[16] of the pole for (x, y, z, t). The scalar potential Ψ is now

$$\Psi=e\frac{\sqrt{c^2-v^2}}{M} \tag{24}$$

and is connected with the potentials A and Φ by the relation[17]

$$\Psi=\sqrt{\Phi^2-A^2} \tag{25}$$

The potentials Ψ, A and Φ, moreover, satisfy the equations

$$\Box\,\Psi=0, \quad div\ A+\frac{1}{c}\frac{\partial\Phi}{\partial t}=0, \quad \Box A=0, \quad \Box\Phi=0. \tag{26}$$

where

$$\Box U\equiv\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \tag{27}$$

Defining the field vectors E and H by means of the equations

$$H=curl\ A, \quad E=-\frac{1}{c}\frac{\partial A}{\partial t}-\nabla\Phi \tag{28}$$

it is found that they satisfy Maxwell's equations

$$curl\ H=\frac{1}{c}\frac{\partial E}{\partial t}, \qquad div\ E=0 \\ curl\ E=-\frac{1}{c}\frac{\partial H}{\partial t}, \quad div\ H=0 \tag{29}$$

and that

$$\Box\ E=0, \quad \Box\ H=0. \tag{30}$$

We also have the relation

$$e^2(E^2-H^2)=\Psi^4 \tag{31}$$

and we may write[18]

$$eE=\Psi^2\frac{s_1-s_2}{1-(s_1\cdot s_2)},\quad eH=\Psi^2\frac{s_2\times s_1}{1-(s_1\cdot s_2)} \tag{32}$$

where s_1 and s_2 are the two real unit vectors which satisfy the relations

$$\begin{aligned} &E+(s\times H)=s(s\,.\,E),\quad H-(s\times E)=s(s\,.\,H)\\ &(s\times E).(s\times H)=0,\qquad (s\times E)^2=(s\times H)^2 \end{aligned} \tag{33}$$

The vector s_1 is in the direction of the radius from the effective position of the electric pole. The vector s_2 does not generally admit of a simple geometrical interpretation, but when the pole moves with uniform velocity along a straight line, s_2 is in the direction of the radius to that position of the pole which could be hit by a bullet travelling from (x, y, z, t) along a straight line with velocity c.

The field vectors may also be expressed in Hargreaves' form[19]

$$H=\nabla\tau\times\nabla\sigma,\quad E=\frac{1}{c}\left[\frac{\partial\sigma}{\partial t}\nabla\tau-\frac{\partial\tau}{\partial t}\nabla\sigma\right] \tag{34}$$

where

$$\sigma=e\frac{K}{M} \tag{35}$$

and

$$K=(x-\xi)\xi''+(y-\eta)\eta''+(z-\zeta)\zeta''+c^2-\xi'^2-\eta'^2-\zeta'^2. \tag{36}$$

These expressions are useful for many purposes. It may be remarked in passing that

$$\frac{1}{c^2}\frac{\partial\sigma}{\partial t}\frac{\partial\tau}{\partial t}-\frac{\partial\sigma}{\partial x}\frac{\partial\tau}{\partial x}-\frac{\partial\sigma}{\partial y}\frac{\partial\tau}{\partial y}-\frac{\partial\sigma}{\partial z}\frac{\partial\tau}{\partial z}=\sqrt{E^2-H^2}=-(s\cdot E)=\frac{\Psi^2}{e} \tag{37}$$

The field vectors may also be expressed in the form

$$E=\nabla\alpha\times\nabla\beta,\quad H=\frac{1}{c}\left[\frac{\partial\alpha}{\partial t}\nabla\beta-\frac{\partial\beta}{\partial t}\nabla\alpha\right] \tag{38}$$

where α and β are certain functions of x, y, z and t. The equations $\alpha=\text{const}$, $\beta=\text{const}$ may be regarded as those of a moving line of electric force.[20] In the case of an electric pole moving in an arbitrary manner along the axis of x, we may write[21]

$$\alpha=c\frac{x-\xi-v(t-\tau)}{c^2(t-\tau)-v(x-\xi)},\quad \beta=\tan^{-1}\frac{y}{z} \tag{39}$$

The lines of electric force have been found in a number of other cases—for instance, in uniform circular motion, in uniform helical motion, in Born's hyperbolic motion and in motions that can be derived from these by means of transformations of certain types.

A line of electric force may be regarded as the locus of a series of points moving along straight lines with velocity c and projected from the moving pole at successive instants, the direction of projection varying according to a law which has been formulated by Leigh Page[22] and the author. If the unit vector s specifies the direction of projection at time τ, the components of s satisfy differential equations which are embodied in the vector equation

$$(c^2-v^2)\frac{ds}{d\tau}=(v-cs)(s\cdot v')+v'[c-(s\cdot v)] \tag{40}$$

This equation may be replaced by a Riccation equation[23] with the single complex dependent variable

$$w=\frac{s_x+i\,s_y}{1+s_z} \tag{41}$$

Many interesting geometrical properties of the lines of force and the associated directions of projection have been given by Leigh Page,[22] F. D. Murnaghan[24] and the author.[25]

The rate of radiation of energy from a moving electric pole may be found by an extension of the method given by Liénard[13], use being now made of the more general tensor components defined in Section II. It is found that the rate of radiation is

$$\frac{4\pi e^2 c}{3}\frac{d^2}{d\tau^2}\left(\frac{1}{c^2-v^2}\right) \tag{42}$$

the contribution of the electromagnetic radiation, as found by Liénard and Larmor[15] being

$$\frac{8\pi e^2 c}{3}\left[\frac{v'^2}{(c^2-v^2)^2}+\frac{(v v')^2}{(c^2-v^2)^3}\right] \tag{43}$$

the remaining part arising from the radiation which depends on the variation of the function Ψ.

In a periodic motion the total radiation of energy per period is zero—a result which is in accordance with the idea of non-radiating orbits that has been used so successfully in atomic theory. The total radiation may also be zero in a non-periodic orbit, if we integrate from apse to apse, an apse being defined as a point on the path where the velocity is a maximum or minimum.

V. The Reflection of Light at a Moving Plane Mirror

Let the equation of the moving mirror be $x=ut$ and let us consider in the first place the effect of the mirror on the field of a moving electric pole whose co-ordinates at time τ are (ξ, η, ζ).

We shall assume that the effect is the same as if an additional field were produced by an electric pole at the image of the moving pole, the image of (ξ, η, ζ, τ) being supposed to be determined by the equations[26]

$$\left.\begin{aligned} \xi^* &= \xi - \frac{2c^2}{c^2-u^2}(\xi - u\tau) & \eta^* &= \eta \\ \tau^* &= \tau - \frac{2u}{c^2-u^2}(\xi - u\tau) & \zeta^* &= \zeta \end{aligned}\right\} \quad (44)$$

which give

$$\xi^* - u\tau^* = -(\xi - u\tau) \quad (45)$$

and so furnish the ordinary laws of reflexion for points that move with the same velocity as the mirror. If (x^*, y^*, z^*, t^*) are derived from (x, y, z, t) by the same set of equations, we have

$$\left.\begin{aligned} &(x^* - \xi^*)^2 + (y^* - \eta^*)^2 + (z^* - \zeta^*)^2 - c^2(t^* - \tau^*)^2 \\ &= (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 - c^2(t - \tau)^2 \end{aligned}\right\} \quad (46)$$

Placing a charge e at (ξ, η, ζ, τ) and a charge $-e$ at $(\xi^*, \eta^*, \zeta^*, \tau^*)$, it is easily verified that, with the notation of Section IV

$$d(\tau^*, \sigma^*) = -d(\tau, \sigma) \quad (47)$$

where the symbol $d(\tau, \sigma)$ is used for an expression of type $d\tau\delta\sigma - d\sigma\delta\tau$, the d's and δ's referring to independent increments.

The above equation may be written in the form

$$\left.\begin{aligned} &H_x^* d(y^*, z^*) + H_y^* d(x^*, x^*) + H_z^*(x^*, y^*) + cE_x^* d(x^*, t^*) + cE_y^* d(y^*, t^*) \\ &+ cE_z^* d(z^*, t^*) + H_x d(y, z) + H_y d(z, x) + H_z d(x, y) + cE_x d(x, t) \\ &+ cE_y d(y, t) + cE_z d(z, t) = 0. \end{aligned}\right\} \quad (48)$$

At points of the moving mirror, however, we have $x^* = x$, $y^* = y$, $z^* = z$, $t^* = t$, and the above equation becomes

$$\left.\begin{aligned} &\overline{H}_x d(y, z) + \overline{H}_y d(z, x) + \overline{H}_z d(x, y) + c\overline{E}_x d(x, t) + c\overline{E}_y d(y, t) \\ &+ c\overline{E}_z d(z, t) = 0 \end{aligned}\right\} \quad (49)$$

where $\overline{E}$ and $\overline{H}$ are the electric and magnetic forces in the total field. This is the condition to be satisfied at the surface of a perfect conductor or perfect reflector. We also have $\overline{\Psi} = 0$. These conditions hold for arbitrary fields, as may be seen by superposition of the fields of a number of moving electric poles.

We may deduce from equation (48) that

$$\left.\begin{aligned} E_x^* &= E_x & H_x^* &= -H_x \\ E_y^* &= -\frac{c^2+u^2}{c^2-u^2}E_y + \frac{2cu}{c^2-u^2}H_z, & H_y^* &= \frac{c^2+u^2}{c^2-u^2}H_y + \frac{2cu}{c^2-u^2}E_z, \\ E_z^* &= -\frac{c^2+u^2}{c^2-u^2}E_z - \frac{2cu}{c^2-u^2}H_y, & H_z^* &= \frac{c^2+u^2}{c^2-u^2}H_z - \frac{2cu}{c^2-u^2}E_y. \end{aligned}\right\} \quad (50)$$

$$\left.\begin{aligned}E_y^* - \frac{u}{c}H_z^* &= -\left(E_y - \frac{u}{c}H_z\right)\\ E_z^* + \frac{u}{c}H_y^* &= -\left(E_z + \frac{u}{c}H_y\right)\\ H_y^* + \frac{u}{c}E_z^* &= H_y + \frac{u}{c}E_z\\ H_z^* + \frac{u}{c}E_y^* &= H_z - \frac{u}{c}E_y\end{aligned}\right\} \tag{51}$$

These equations give us the usual boundary conditions at the surface of a perfect conductor, viz.,

$$\overline{H}_x = 0,\ \overline{E}_y = \frac{u}{c}\overline{H}_z,\ E_z + \frac{u}{c}\overline{H}_y = 0.$$

These equations and the equation $\Psi = 0$ indicate that in the total field there is no flow of energy across the surface of the mirror.

We also have

$$E_x^* d(y^*, z^*) + E_y^* d(z^*, x^*) + E_z^* d(x^*, y^*) - cH_x^* d(x^*, t^*) - cH_y^*(y^*, t^*) \\ - cH_z^* d(z^*, t^*) = E_x d(y, z) + E_y d(z, x) + E_z d(x, y) - cH_x d(x, t) \\ - cH_y d(y, t) - cH_z d(z, t) \ (51)$$

or

$$d(\alpha^*, \beta^*) = d(\alpha, \beta). \tag{52}$$

This equation may be interpreted to mean that the lines of electric force of the pole (ξ^*, η^*, ζ^*, τ^*) are the images in the moving mirror of the lines of electric force of the moving pole (ξ, η, ζ, τ). It is important to notice that if the point (ξ, η, ζ, τ) moves with a velocity less than c, the point (ξ^*, η^*, ζ^*, τ^*) does also. The image in the mirror of a stationary observer is an observer moving with velocity

$$v = \frac{2c^2 u}{c^2 + u^2} \tag{53}$$

in the direction of the axis of x. Referred to the stationary axes this observer will suffer from the Lorentz-Fitzgerald contraction

$$\sqrt{1 - \frac{v^2}{c^2}} : 1.$$

as is easily seen from the equations.

All observations made by the moving observer may be regarded as images in the mirror of corresponding observation's made by the stationary observer and we may deduce the whole theory of the relativity transformation from reflections in moving mirrors.

It is important to notice that a mirror and an associated set of moving points and images reflect into a mirror and an associated set of moving points and images provided all mirrors move uniformly. This result is of importance in the theory of the Michelson and Morley experiment, for the essential feature of the arrangement of mirrors is that *A* is the image of B in *C*.

This implies that successive reflections in *C* and *A* are equivalent to successive reflections in *B* and *C*. Since *A* is the image of *B* in *C* whatever be the direction of the Earth's uniform motion,* the explanation of the negative result of the experiment is, perhaps, clear, for any instantaneous event which happens at a stationary or moving point *P*, which we suppose to be a source of light, has the same instantaneous image whether the light is reflected first in *C* and then in *A* or first in *B* and then in *C*. This is true for different positions of the apparatus relative to the Earth's direction of motion and so the conditions for interference are the same in all cases.

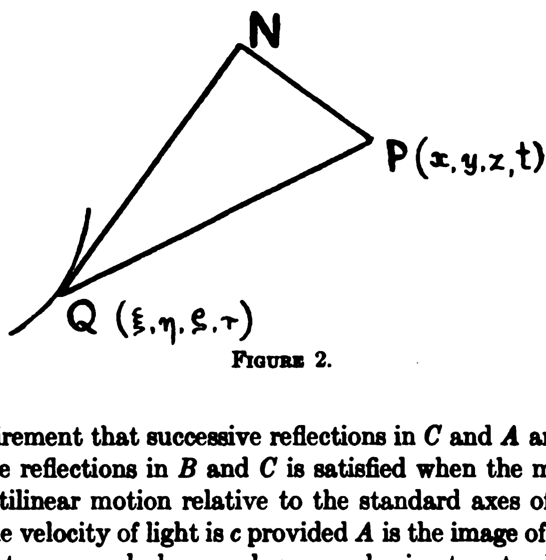

FIGURE 2.

The requirement that successive reflections in *C* and *A* are equivalent to successive reflections in *B* and *C* is satisfied when the mirrors are in uniform rectilinear motion relative to the standard axes of coordinates for which the velocity of light is *c* provided *A* is the image of *B* in *C* when measurements are made by an observer who is at rest relative to the mirrors. It is not necessary for the mirrors *A* and *B* to be perpendicular; it is simply sufficient that *C* should bisect the angle between the planes *A* and *B*, *when the angles are measured by an observer moving with the same velocity as the mirrors*. The source of light *P* can have any motion whatever.

* A. Righi, Comptes Rendus, t. 168 (1919), p. 837, t. 170, pp. 497, 1550, concludes that a nul result is to be expected for any orientation of the apparatus relative to the earth's direction of translational motion.

VI. Behaviour of Electrical Quantities Under a Relativity Transformation

In order that a clear idea may be obtained of the mathematical properties of the quantities that have been introduced we shall indicate the behaviour of these quantities under the group of Lorentzian transformations,[27] a typical transformation of the group being

$$x^1 = \frac{x - wt}{\sqrt{1 - \frac{w^2}{c^2}}}, \quad t^1 = \frac{t - \frac{wx}{c^2}}{\sqrt{1 - \frac{w^2}{c^2}}}, \quad y^1 = y, \quad z^1 = z. \tag{54}$$

The quantities $\Psi, \rho\sqrt{1 - \frac{v^2}{c^2}}, E^2 - H^2, (E \cdot H)$ *and* $\Phi^2 - A^2$ are invariants. So also are

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\psi}{\partial t}\right)^2$$

$$\rho[c\Phi - (A.v)]$$

$$(x - \xi)^2 + (y - y)^2 + (z - \zeta)^2 - c^2(t - \tau)^2$$

$$dx^2 + dy^2 + dz^2 - c^2dt^2.$$

$$d(\alpha, \beta), \quad d(\sigma, \tau), \quad d(x, y, z, t)$$

$$A_x dx + A_y dy + A_z dz - c\Phi dt$$

$$H_x d(y, z) + H_y d(z, x) + H_z d(x, y) + cE_x d(x, t) + cE_y d(y, t) + cE_z d(z, t)$$

$$E_x d(y, z) + E_y d(z, x) + E_z d(x, y) - cH_x d(x, t) - cH_y d(y, t) - cH_z d(z, t)$$

The vectors v, cs_1 and cs_2 transform like velocities. If we use the Poincaré-Minkowski representation of a space-time point (x, y, z, t) by a point with rectangular co-ordinates (x, y, z, ict) in a space of four dimensions S_4, a moving point is represented by a curve or *world line* in S_4 and we may introduce the direction cosines (w_1, w_2, w_3, w_4) of the tangent at a point of this curve. These direction cosines are connected with the velocity of the moving point at the corresponding instant by the equations.[28]

$$iw_1 = \frac{v_x}{(c^2 - v^2)^{\frac{1}{2}}}, \quad iw_2 = \frac{v_y}{(c^2 - v^2)^{\frac{1}{2}}}, \quad iw_3 = \frac{v_z}{(c^2 - v^2)^{\frac{1}{2}}}, \quad iw_4 = \frac{ic}{(c^2 - v^2)^{\frac{1}{2}}} \tag{55}$$

and of course

$$w_1^2 + w_1^2 + w_3^2 + w_4^2 = 1 \tag{56}$$

With the aid of these direction cosines Minkowski[21] was able to express the potentials of a moving point charge in the symmetrical form

$$A_x = \frac{ew_1}{R_w}, \quad A_y = \frac{ew_2}{R_w}, \quad A_z = \frac{ew_3}{R_w}, \quad i\Phi = \frac{ew_4}{R_w} \tag{57}$$

where

$$R_w = w_1(x - \xi) + w_2(y - \eta) + w_3(z - \zeta) + icw_4(t - \tau) \tag{58}$$

is the projection of the radius QP on the tangent QN to the world line.

We also have

$$i\Psi = \frac{e}{R_w} \tag{59}$$

The function Ψ is in fact analogous to a function used by M. Abraham[21] in his theory of gravitation.

It was pointed out by E. B. Wilson[21] and G. N. Lewis and also later by Megh Nad Saha[22] that the potentials can be written in the form

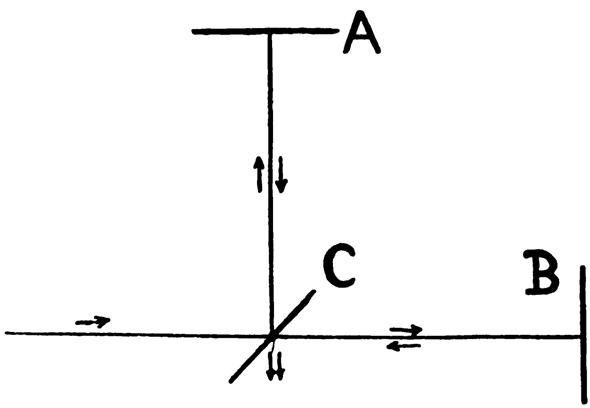

FIGURE 3.

$$A_x = \frac{iew_1}{P}, \quad A_y = \frac{iew_2}{P}, \quad A_z = \frac{iew_3}{P}, \quad i\Phi = \frac{iew_4}{P} \tag{60}$$

where P is the perpendicular distance of P from the tangent at Q, so that

$$P^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 - c^2(t-\tau)^2 - [w_1(x-\xi) + w_2(y-\eta) + w_3(z-\zeta) + icw_4(t-\tau)]^2$$

It is clear that $P = iR_w$ on account of the relation (23).

The field vectors E and H may also be given a geometrical interpretation. To see this let Q, R and S be three consecutive points on the world line of an electric pole, and let Q be the point in S_4 which corresponds to the effective position of the pole for a space-time point (x, y, z, t) represented in S_4 by P. The line PQ is a minimal or isotropic line, i. e. a line of zero length.

Consider the sphere which passes through $PQRS$. It will have a tangent plane at P and in this tangent plane there will be two isotropic lines, viz PQ and a line PT. This line PT has a direction corresponding to a velocity cs_2 while QP has a direction corresponding to the velocity cs_1. The unit vectors s_1 and s_2 and the scalar Ψ needed for the expressions (32) for E and H thus admit of a geometrical interpretation.

The six components of E and iH can be regarded as $\frac{\Psi^2}{e}$ times the direction cosines of the tangent plane at P to the sphere just mentioned.

In the case of uniform rectilinear motion the sphere reduces to a plane and it can be shown that the field vectors at (x, y, z, t) can be expressed in terms of the positional co-ordinates of the electric pole at any time. if these are (ξ, η, ζ) at time τ we may write[22]

$$E_x - i\,H_x = -e(c^2\tau^2 - \xi^2 - \eta^2 - \zeta^2)\frac{X}{R^3} \tag{62}$$

where

$$X = c(x\tau - \xi t) + i(y\zeta - z\eta), \qquad R^2 = X^2 + Y^2 + Z^2 \tag{63}$$

VII. Fields With Singularities of a Complex Nature Which Move With Velocities Less Than C.

The general type of electromagnetic field with an isolated simple singularity or pole which moves with a velocity less than c may be specified by means of the equations

$$H + iE \equiv Q = curl\ C - \frac{i}{c}\frac{\partial C}{\partial t} - i\nabla\Gamma \tag{64}$$

$$C = -\frac{m}{4\pi}\frac{v}{M}, \qquad \Gamma = -\frac{m}{4\pi}\frac{c}{M} \tag{65}$$

where $m = h + ie$ is a complex constant, $iC = B + iA$, $i\Gamma = \Omega + i\Phi$. The potentials B and Ω are the potentials of magnetic type while A and Φ are the potentials of electric type. Equating the real and imaginary parts in equation (64) we have

$$H = curl\ A - \nabla\Omega - \frac{1}{c}\frac{\partial B}{\partial t}$$

$$E = -curl\ B - \nabla\Phi - \frac{1}{c}\frac{\partial A}{\partial t}$$

We may call e the electric charge and h the magnetic charge associated with the moving pole. When $h = o$ we obtain the formulae of Liénard.

It should be noticed that

$$div\ C + \frac{1}{c}\frac{\partial\Gamma}{\partial t} = 0, \qquad div\ A + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0, \qquad div\ B + \frac{1}{c}\frac{\partial\Omega}{\partial t} = 0 \tag{66}$$

We may also write

$$\Pi = -\frac{m}{4\pi}\frac{\sqrt{c^2 - v^2}}{M}, \qquad i\Pi = \Lambda + i\Psi \tag{67}$$

and so introduce a function Λ which is the magnetic analogue of Ψ.

The mathematical analysis for fields with complex singularities may be developed in a simple manner by using contour integrals of type

$$\frac{f(\tau)}{M} = \frac{i}{\pi}\int_c \frac{f(s)ds}{[x-\xi(s)]^2 + [y-\eta(s)]^2 + [z-\zeta(s)]^2 - c^2(t-s)^2} \tag{68}$$

as recommended by A. W. Conway,[24] G. Herlotz[25] and others. The proof that our potentials C, Γ, Π are wave-functions satisfying $\Box C = 0$, $\Box\Gamma = 0$, $\Box\Pi = 0$, $div\ C + \frac{1}{c}\frac{\partial\Gamma}{\partial t} = 0$ is immediate and it can also be shown at once that if f is an arbitrary function of τ, we have the identity

$$\frac{f'(\tau)}{M}=\frac{\partial}{\partial x}\left(\frac{f\xi'}{M}\right)+\frac{\partial}{\partial y}\left(\frac{f\eta'}{M}\right)+\frac{\partial}{\partial z}\left(\frac{f\zeta'}{M}\right)+\frac{\partial}{\partial t}\left(\frac{f}{M}\right) \tag{69}$$

To obtain fields with isolated moving singularities of higher order we consider a neighboring pole whose position at time τ is given by the equations

$$\begin{aligned}\xi_1(\tau)&=\xi(\tau)+\theta\alpha(\tau)+\frac{1}{2}\theta^2\lambda(\tau)+\frac{1}{6}\theta^3\Phi(\tau)+\\ \eta_1(\tau)&=\eta(\tau)+\theta\beta(\tau)+\frac{1}{2}\theta^2\mu(\tau)+\frac{1}{6}\theta^3\psi(\tau)+\\ \zeta_1(\tau)&=\zeta(\tau)+\theta\gamma(\tau)+\frac{1}{2}\theta^2\nu(\tau)+\frac{1}{6}\theta^3\chi(\tau)+\end{aligned} \tag{70}$$

Forming potentials C_1, Γ_1, Π_1 in the same way as before and expressing them as contour integrals of Conway's type, we find on expanding them in ascending powers of the small constant θ and writing $C^*=C-C_1$, $\Gamma^*=\Gamma-\Gamma_1$, $\Pi^*=\Pi-\Pi_1$, that

$$\begin{aligned}\frac{4\pi}{m}C_x^*=\theta&\left[\frac{\alpha'}{M}-\frac{\partial}{\partial x}\left(\frac{\alpha\xi'}{M}\right)-\frac{\partial}{\partial y}\left(\frac{\beta\xi'}{M}\right)-\frac{\partial}{\partial z}\left(\frac{\gamma\xi'}{M}\right)\right]\\
&+\frac{1}{2}\theta^2\left[\frac{\lambda'}{M}-2\frac{\partial}{\partial x}\left(\frac{\alpha'\alpha}{M}\right)-2\frac{\partial}{\partial y}\left(\frac{\alpha'\beta}{M}\right)-2\frac{\partial}{\partial z}\left(\frac{\alpha'\gamma}{M}\right)-\frac{\partial}{\partial x}\left(\frac{\lambda\xi'}{M}\right)-\frac{\partial}{\partial y}\left(\frac{\mu\xi'}{M}\right)\right.\\
&-\frac{\partial}{\partial z}\left(\frac{\nu\xi'}{M}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\alpha^2\xi^1}{M}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\beta^2\xi'}{M}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{\gamma^2\xi'}{M}\right)+2\frac{\partial^2}{\partial y\partial z}\left(\frac{\beta\gamma\xi'}{M}\right)\\
&\left.+2\frac{\partial^2}{\partial z\partial x}\left(\frac{\gamma\alpha\xi'}{M}\right)+2\frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha\beta\xi'}{M}\right)\right]+\frac{1}{6}\theta^3\left[\frac{\phi'}{M}-3\frac{6}{\partial x}\left(\frac{\lambda'\alpha}{M}\right)\right.\\
&-3\frac{\partial}{\partial y}\left(\frac{\gamma'\beta}{M}\right)-3\frac{\partial}{\partial z}\left(\frac{\lambda'\gamma}{M}\right)-3\frac{\partial}{\partial x}\left(\frac{\alpha'\lambda}{M}\right)-3\frac{\partial}{\partial y}\left(\frac{\alpha'\mu}{M}\right)-3\frac{\partial}{\partial z}\left(\frac{\alpha'\nu}{M}\right)\\
&-\frac{\partial}{\partial x}\left(\frac{\phi\xi'}{M}\right)-\frac{\partial}{\partial y}\left(\frac{\psi\xi'}{M}\right)-\frac{\partial}{\partial z}\left(\frac{\chi\xi'}{M}\right)+3\frac{\partial^2}{\partial x^2}\left(\frac{\alpha'\alpha^2}{M}\right)+3\frac{\partial^2}{\partial y^2}\left(\frac{\alpha'\beta^2}{M}\right)\\
&+3\frac{\partial^2}{\partial z^2}\left(\frac{\alpha'\gamma^2}{M}\right)+6\frac{\partial^2}{\partial y\partial z}\left(\frac{\alpha'\beta\gamma}{M}\right)+6\frac{\partial^2}{\partial z\partial x}\left(\frac{\alpha'\gamma\alpha}{M}\right)+6\frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha'\alpha\beta}{M}\right)\\
&+3\frac{\partial^2}{\partial x^2}\left(\frac{\xi'\alpha\lambda}{M}\right)+3\frac{\partial^2}{\partial y^2}\left(\frac{\xi'\beta\mu}{M}\right)+3\frac{\partial^2}{\partial z^2}\left(\frac{\xi'\gamma\nu}{M}\right)+3\frac{\partial^2}{\partial y\partial z}\left[\frac{\xi'(\beta\nu+\gamma\mu)}{M}\right]\\
&+3\frac{\partial^2}{\partial z\partial x}\left\{\frac{\xi'(\gamma\lambda+\alpha\nu)}{M}\right\}+3\frac{\partial^2}{\partial x\partial y}\left\{\frac{\xi'(\alpha\mu+\beta\lambda)}{M}\right\}-\frac{\partial^3}{\partial x^3}\left\{\frac{\xi'\alpha^3}{M}\right\}\\
&-\frac{\partial^3}{\partial y^3}\left\{\frac{\xi'\beta^3}{M}\right\}-\frac{\partial^3}{\partial z^3}\left\{\frac{\xi'\gamma^3}{M}\right\}-3\frac{\partial^3}{\partial y^2\partial z}\left\{\frac{\xi'\beta^2\gamma}{M}\right\}-3\frac{\partial^3}{\partial y\partial z^2}\left\{\frac{\xi'\beta\gamma^2}{M}\right\}\\
&-3\frac{\partial^3}{\partial z^2\partial x}\left\{\frac{\xi'\gamma^2\alpha}{M}\right\}-3\frac{\partial^3}{\partial z\partial x^2}\left\{\frac{\xi'\alpha^2\gamma}{M}\right\}-3\frac{\partial'}{\partial x^2\partial y}\left\{\frac{\xi'\alpha^2\beta}{M}\right\}\\
&-3\frac{\partial^3}{\partial x\partial y^2}\left\{\frac{\xi'\alpha\beta^2}{M}\right\}-6\frac{\partial^3}{\partial x\partial y\partial z}\left\{\frac{\xi'\alpha\beta\gamma}{M}\right\}\end{aligned} \tag{71}$$

with corresponding expressions for Γ^* and Π^*. Making use of the identity (69) we find that we may write

$$\frac{4\pi}{m}C^*=\frac{\partial L}{\partial t}+curl\ N, \tag{72}$$

where

$$\begin{aligned}L_x=\theta\frac{\alpha}{M}+\frac{1}{2}\theta^2\left[\frac{\lambda}{M}-\frac{\partial}{\partial x}\left(\frac{\alpha^2}{M}\right)-\frac{\partial}{\partial y}\left(\frac{\beta\alpha}{M}\right)-\frac{\partial}{\partial z}\left(\frac{\gamma\alpha}{M}\right)\right]+\frac{1}{6}\theta^3\left[\frac{\phi}{M}\right.\\ -3\frac{\partial}{\partial x}\left(\frac{\alpha\lambda}{M}\right)-3\frac{\partial}{\partial y}\left(\frac{\beta\lambda}{M}\right)-3\frac{\partial}{\partial z}\left(\frac{\gamma\lambda}{M}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\alpha^3}{M}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\alpha\beta^2}{M}\right)\\ \left.+\frac{\partial^2}{\partial z^2}\left(\frac{\alpha\gamma^2}{M}\right)+2\frac{\partial^2}{\partial y\partial z}\left(\frac{\alpha\beta\gamma}{M}\right)+2\frac{\partial^2}{\partial z\partial x}\left(\frac{\alpha^2\gamma}{M}\right)+2\frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha^2\beta}{M}\right)\right]\end{aligned} \tag{73}$$

$$\begin{aligned}N_x=\theta\frac{\beta\zeta'-\gamma\eta'}{M}+\frac{1}{2}\theta^2\left[\frac{\mu\zeta'-\nu\eta'}{M}+\frac{\beta\gamma'-\beta'\gamma}{M}+\frac{\partial}{\partial x}\left\{\frac{\alpha(\eta'\gamma-\zeta'\beta)}{M}\right\}\right.\\ \left.+\frac{\partial}{\partial y}\left\{\frac{\beta(\eta'\gamma-\zeta'\beta)}{M}\right\}+\frac{\partial}{\partial z}\left\{\frac{\gamma(\eta'\gamma-\zeta'\beta)}{M}\right\}\right]+\frac{1}{6}\theta^3\left[\frac{\psi\zeta'-\chi\eta'}{M}\right.\\ +3\frac{\mu\gamma'-\nu\beta'}{M}+3\frac{\partial}{\partial x}\left\{\frac{\alpha(\eta'\nu-\zeta'\mu)}{M}\right\}+3\frac{\partial}{\partial y}\left\{\frac{\beta(\eta'\nu-\zeta'\mu)}{M}\right\}\\ +3\frac{\partial}{\partial z}\left\{\frac{\gamma(\eta'\nu-\zeta'\mu)}{M}\right\}+2\frac{\partial}{\partial x}\left\{\frac{\alpha(\beta'\gamma-\beta\gamma')}{M}\right\}+2\frac{\partial}{\partial y}\left\{\frac{\beta(\beta'\gamma-\beta\gamma')}{M}\right\}\\ +2\frac{\partial}{\partial z}\left\{\frac{\gamma(\beta'\gamma-\beta\gamma')}{M}\right\}+\frac{\partial^2}{\partial x^2}\left\{\frac{\alpha^2(\beta\zeta'-\gamma\eta')}{M}\right\}+\frac{\partial^2}{\partial y^2}\left\{\frac{\beta^2(\beta\zeta'-\gamma\eta')}{M}\right\}\\ +\frac{\partial^2}{\partial z^2}\left\{\frac{\gamma^2(\beta\zeta'-\gamma\eta')}{M}\right\}+2\frac{\partial^2}{\partial y\partial z}\left\{\frac{\beta\gamma(\beta\zeta'-\gamma\eta')}{M}\right\}+2\frac{\partial^2}{\partial z\partial x}\left\{\frac{\gamma\alpha(\beta\zeta'-\gamma\eta')}{M}\right\}\\ \left.+2\frac{\partial^2}{\partial x\partial y}\left\{\frac{\alpha\beta(\beta\zeta'-\gamma\eta')}{M}\right\}\right]+\end{aligned} \tag{74}$$

We may also write*

$$\frac{4\pi}{m}\Gamma^*=-c\ div\ L \tag{75}$$

$$\begin{aligned}\Pi^*=\theta\left[\frac{\partial}{\partial x}\left(\frac{\alpha\kappa}{M}\right)+\frac{\partial}{\partial y}\left(\frac{\beta\kappa}{M}\right)+\frac{\partial}{\partial z}\left(\frac{\gamma\kappa}{M}\right)+\frac{\xi'\alpha'+\eta'\beta'+\zeta'\gamma'}{\kappa M}\right]+\frac{1}{2}\theta^2\left[\frac{\partial}{\partial x}\left(\frac{\lambda\kappa}{M}\right)\right.\\ +\frac{\partial}{\partial y}\left(\frac{\mu\kappa}{M}\right)+\frac{\partial}{\partial z}\left(\frac{\nu\kappa}{M}\right)-2\frac{\partial}{\partial x}\left(\frac{\alpha\sigma}{\kappa M}\right)-2\frac{\partial}{\partial y}\left(\frac{\beta\sigma}{\kappa M}\right)-2\frac{\partial}{\partial z}\left(\frac{\gamma\sigma}{\kappa M}\right)-\frac{\partial^2}{\partial x^2}\left(\frac{\alpha^2\kappa}{M}\right)\\ -\frac{\partial^2}{\partial y^2}\left(\frac{\beta^2\kappa}{M}\right)-\frac{\partial^2}{\partial z^2}\left(\frac{\gamma^2\kappa}{M}\right)-2\frac{\partial^2}{\partial y\partial z}\left(\frac{\beta\gamma\kappa}{M}\right)-2\frac{\partial^2}{\partial z\partial x}\left(\frac{\gamma\alpha\kappa}{M}\right)-2\frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha\beta\kappa}{M}\right)\\ \left.+\frac{\omega}{\kappa M}+\frac{\sigma^2}{\kappa^3 M}\right]+\frac{1}{6}\theta^3\left[\frac{\partial^3}{\partial x^3}\left(\frac{\alpha^2\kappa}{M}\right)+\frac{\partial^2}{\partial y^3}\left(\frac{\beta^2\kappa}{M}\right)+\frac{\partial^3}{\partial z^3}\left(\frac{\gamma^3\kappa}{M}\right)+3\frac{\partial^3}{\partial y^2\partial z}\left(\frac{\beta^2\gamma\kappa}{M}\right)\right.\end{aligned}$$

* These cumbersome expressions are needed for a calculation of the radiation from the number of moving electric poles. The results of the calculation will be published later.

$$+3\frac{\partial^3}{\partial y\partial z^2}\left(\frac{\beta\gamma^2\kappa}{M}\right)+3\frac{\partial^3}{\partial z^2\partial x}\left(\frac{\gamma^2\alpha\kappa}{M}\right)+3\frac{\partial^3}{\partial z\partial x^2}\left(\frac{\gamma\alpha^2\kappa}{M}\right)+3\frac{\partial^3}{\partial x^2\partial y}\left(\frac{\alpha^2\beta\kappa}{M}\right)$$
$$+3\frac{\partial^3}{\partial x\partial y^2}\left(\frac{\alpha\beta^2\kappa}{M}\right)+6\frac{\partial^3}{\partial x\partial y\partial z}\left(\frac{\gamma\alpha\beta\kappa}{M}\right)-3\frac{\partial^2}{\partial x^2}\left(\frac{\alpha\lambda\kappa}{M}\right)-3\frac{\partial^2}{\partial y^2}\left(\frac{\beta\mu\kappa}{M}\right)$$
$$-3\frac{\partial^2}{\partial z^2}\left(\frac{\gamma\nu\kappa}{M}\right)-3\frac{\partial^2}{\partial y\partial z}\left\{\frac{(\beta\nu+\gamma\mu)\kappa}{M}\right\}-3\frac{\partial^2}{\partial z\partial x}\left\{\frac{(\gamma\lambda+\alpha\nu)\kappa}{M}\right\}$$
$$-3\frac{\partial^2}{\partial x\partial y}\left\{\frac{(\alpha\mu+\beta\lambda)\kappa}{M}\right\}+3\frac{\partial^2}{\partial x^2}\left(\frac{\alpha^2\sigma}{\kappa M}\right)+3\frac{\partial^2}{\partial y^2}\left(\frac{\beta^2\sigma}{\kappa M}\right)+3\frac{\partial^2}{\partial z^2}\left(\frac{\gamma^2\sigma}{\kappa M}\right)$$
$$+6\frac{\partial^2}{\partial y\partial z}\left(\frac{\beta\gamma\sigma}{\kappa M}\right)+6\frac{\partial^2}{\partial z\partial x}\left(\frac{\gamma\alpha\sigma}{\kappa M}\right)+6\frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha\beta\sigma}{\kappa M}\right)+\frac{\partial}{\partial x}\left(\frac{\phi\kappa}{M}\right)+\frac{\partial}{\partial y}\left(\frac{\psi\kappa}{M}\right)$$
$$+\frac{\partial}{\partial z}\left(\frac{\chi\kappa}{M}\right)-3\frac{\partial}{\partial x}\left(\frac{\lambda\sigma}{\kappa M}\right)-3\frac{\partial}{\partial y}\left(\frac{\mu\sigma}{\kappa M}\right)-3\frac{\partial}{\partial z}\left(\frac{\nu\sigma}{\kappa M}\right)-3\frac{\partial}{\partial x}\left(\frac{\omega\alpha}{\kappa M}\right)-3\frac{\partial}{\partial y}\left(\frac{\omega\beta}{\kappa M}\right)$$
$$-3\frac{\partial}{\partial z}\left(\frac{\omega\gamma}{\kappa M}\right)-3\frac{\partial}{\partial x}\left(\frac{\alpha\sigma^2}{\kappa^3 M}\right)-3\frac{\partial}{\partial y}\left(\frac{\beta\sigma^2}{\kappa^3 M}\right)-3\frac{\partial}{\partial z}\left(\frac{\gamma\sigma^2}{\kappa^3 M}\right)+3\frac{\sigma^3}{\kappa^5 M}+\frac{3\sigma\omega}{\kappa^3 M}$$
$$\left.+\frac{3(\alpha'\lambda'+\beta'\mu'+\gamma'\nu')+\xi'\phi'+\eta'\psi'+\zeta'\chi'}{\kappa M}\right]+\cdot\cdot\cdot\cdot \tag{76}$$

where

$$\begin{aligned}\sigma&=\xi'\alpha'+\eta'\beta'+\zeta'\gamma',\kappa=\sqrt{c^2-v^2}\\ \omega&=\alpha'^2+\beta'^2+\gamma'^2+\xi'\lambda'+\eta'\mu'+\zeta'\nu'.\end{aligned} \tag{77}$$

Since L and N satisfy $\Box L=0$. $\Box N=0$, it is easy to verify that we have the identities

$$curl\ C^*=-\frac{i}{c}\frac{\partial D}{\partial t}-i\nabla\Sigma \tag{78}$$

$$\frac{1}{c}\frac{\partial C^*}{\partial t}+\nabla\Gamma^*=i\ curl\ D \tag{79}$$

where

$$\left.\begin{aligned}D&=\frac{m}{4\pi}\left[ic\ curl\ \mathrm{L}-\frac{i}{c}\frac{\partial N}{\partial t}\right]\\ \Sigma&=\frac{im}{4\pi}\ div\ N\end{aligned}\right\} \tag{80}$$

Hence, if we write

$$\left.\begin{aligned}C^*+D=C_0,\quad \Gamma^*+\Sigma=\Gamma_0\\ \frac{m}{4\pi}(-cL+iN)=G_0\end{aligned}\right\} \tag{81}$$

we may specify an electromagnetic field by means of the equations

$$Q_0=H_0+iE_0=curl\ C_0\equiv-\frac{i}{c}\left[\frac{\partial C_0}{\partial t}+c\nabla\Gamma_0\right] \tag{82}$$

$$C_0=\frac{1}{c}\frac{\partial G_0}{\partial t}+i\ curl\ G_0,\quad \Gamma_0=-div\ G_0 \tag{83}$$

where G_0 is a generalized Hertzian vector which satisfies the wave-equation $\Box G_0=0$. The field thus represented is really the difference of the fields of the two neighboring poles.

We must now examine in detail the fields obtained by separating the different powers of θ. Let us write

$$G_0=\theta G_1+\frac{1}{2}\theta^2G_2+\frac{1}{6}\theta^3G_3+\cdots \tag{84}$$

and use a to denote the vector with components (α, β, γ), then we may regard the field derived from G_1, as the electromagnetic field of the combination of a moving electric doublet of electric moment ae and a moving magnetic doublet of magnetic moment ah. The combination may be called an *electromagnetic doublet*, or dipole.

The field derived from G_2 consists of two parts, one of which is the field of an electromagnetic doublet, whose complex moment $g_1=q_1+ip_1$ is given by

$$g_1+\frac{i}{c}[v\times g_1]=m\left\{b+\frac{i}{c}[v\times b]+\frac{i}{c}[a'\times a]\right\} \tag{85}$$

where b denotes the vector with components (λ, μ, ν) and a' the vector with components $(\alpha', \beta', \gamma')$. The other part of the field may be regarded as the field of a combination of quadrupoles whose physical properties are described by means of a diad with components

$$\begin{matrix}\alpha g_x & \beta g_x & \gamma g_x\\ \alpha g_y & \beta g_y & \gamma g_y\\ \alpha g_z & \beta g_z & \gamma g_z\end{matrix}$$

g being the complex moment ma.

The field derived from G_3 consists of three parts which may be regarded as the fields of an electromagnetic dipole, a combination of quadrupoles and a combination of octupoles. The moment of the dipole is g_2 where

$$g_2+\frac{i}{c}[v\times g_2]=m\left\{k+\frac{i}{c}[a'\times b]\right\} \tag{86}$$

k being the vector with component (ϕ, ψ, χ). The diad specifying the properties of the combination of quadrupoles has the components

$$\begin{matrix}\alpha(m\lambda+g_{1x}) & \beta(m\lambda+g_{1x}) & \gamma(m\lambda+g_{1x})\\ \alpha(m\mu+g_{1y}) & \beta(m\mu+g_{1y}) & \gamma(m\mu+g_{1y})\\ \alpha(m\nu+g_{1z}) & \beta(m\nu+g_{1z}) & \gamma(m\nu+g_{1z})\end{matrix}$$

The physical properties of the combination of octupoles may be specified by means of a triad with 27 components which may be represented by the scheme

$$g_x\begin{bmatrix}\alpha^2 & \alpha\beta & \alpha\gamma\\ \alpha\beta & \beta^2 & \beta\gamma\\ \alpha\gamma & \beta\gamma & \gamma^2\end{bmatrix},\ g_y\begin{bmatrix}\alpha^2, & \alpha\beta, & \alpha\gamma\\ \alpha\beta, & \beta^2, & \beta\gamma\\ \alpha\gamma, & \beta\gamma, & \gamma^2\end{bmatrix},\ g_z\begin{bmatrix}\alpha^2 & \alpha\beta & \alpha\gamma\\ \alpha\beta & \beta^2 & \beta\gamma\\ \alpha\gamma & \beta\gamma & \gamma^2\end{bmatrix}$$

It should be noticed that on account of the velocity of the singularity (ξ, η, ζ) an electromagnetic moment g_s enters into the expressions for the potentials of different orders in the form of a derived vector

$$f_s = g_s + \frac{\dot{i}}{c}[v \times g_s] \tag{87}$$

and the effective diads and triads are obtained by replacing the g's in the above expressions by f's. If p and q are electric and magnetic moments respectively, we can say that the *effective moments* are

$$p + \frac{\dot{i}}{c}[v \times q] \text{ and } q - \frac{\dot{i}}{c}[v \times p]$$

respectively.

It should be noticed that in the case of the field derived from G_2 there is a magnetic moment even when $h = 0$, i. e., when there are no magnetic poles.

According to the electron theory the origin of magnetism is accounted for in this way* but according to the recent theory of Ewing[36] it is necessary to consider electrical systems equivalent to magnetic quadrupoles to account for the magnetic properties of iron. Such systems cannot be specified in this point singularity method by means of Hertzian vectors of types G_1 and G_2; it is necessary to use Hertzian vectors of higher order and so the rather cumbersome expression for G_3 is not without interest. It may be remarked that our definition of the magnetic moment differs slightly from the one usually adopted, but the present definition seems the most natural one as it preserves the usual symmetry; it also gives the usual definition when $v = 0$.

When $\dot{\xi} = \dot{\eta} = \dot{\zeta} = 0$ for all values of τ so that the primary singularity is stationary, our results take a simpler form. We now have $M = -cr$, where r is the distance from the origin. All quantities such as $\alpha, \beta, \gamma, \lambda, \mu, \nu, \theta, \phi, \psi$ and their derivatives are functions of $t - \frac{r}{c}$. We thus find that

$$\begin{aligned} -cL_x = \theta\frac{\alpha}{r} &+ \frac{1}{2}\theta^2\left[\frac{\lambda}{r} - \frac{\partial}{\partial x}\left(\frac{\alpha^2}{r}\right) - \frac{\partial}{\partial y}\left(\frac{\beta\alpha}{r}\right) - \frac{\partial}{\partial z}\left(\frac{\gamma\alpha}{r}\right)\right] \\ &+ \frac{1}{6}\theta^3\left[\frac{\phi}{r} - 3\frac{\partial}{\partial x}\left(\frac{\alpha\lambda}{r}\right) - 3\frac{\partial}{\partial y}\left(\frac{\beta\lambda}{r}\right) - 3\frac{\partial}{\partial z}\left(\frac{\gamma\lambda}{r}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\alpha^3}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\alpha\beta^2}{r}\right)\right. \\ &\left. + \frac{\partial^2}{\partial z^2}\left(\frac{\alpha\gamma^2}{r}\right) + 2\frac{\partial^2}{\partial y \partial z}\left(\frac{\alpha\beta\gamma}{r}\right) + 2\frac{\partial^2}{\partial z \partial x}\left(\frac{\alpha^2\gamma}{r}\right) + 2\frac{\partial^2}{\partial x \partial y}\left(\frac{\alpha^2\beta}{r}\right)\right] + \cdots \end{aligned} \tag{88}$$

$$\begin{aligned} -cN_x = \frac{1}{2}\theta^2\left[\frac{\beta\gamma' - \beta'\gamma}{r}\right] &+ \frac{1}{6}\theta^3\left[3\frac{\mu\gamma' - \nu\beta'}{r} + 2\frac{\partial}{\partial x}\left\{\frac{\alpha(\beta'\gamma - \beta\gamma')}{r}\right\}\right. \\ &\left. + 2\frac{\partial}{\partial y}\left\{\frac{\beta(\beta'\gamma - \beta\gamma')}{r}\right\} + 2\frac{\partial}{\partial z}\left\{\frac{\gamma(\beta'\gamma - \beta\gamma')}{r}\right\}\right] \end{aligned} \tag{89}$$

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* See, for instance, H. A. Lorentz, "Theory of Electrons," and "Vorträge über kinetische Theorie der Materie," Leipsig, Teubner (1914); H. J. van Leeuwen, Diss. Leiden, (1919).

$$-\Pi=\theta\left[\frac{\partial}{\partial x}\left(\frac{\alpha}{r}\right)+\frac{\partial}{\partial y}\left(\frac{\beta}{r}\right)+\frac{\partial}{\partial z}\left(\frac{\gamma}{r}\right)\right]+\frac{1}{2}\theta^2\left[\frac{\partial}{\partial x}\left(\frac{\lambda}{r}\right)+\frac{\partial}{\partial y}\left(\frac{\mu}{r}\right)+\frac{\partial}{\partial z}\left(\frac{\nu}{r}\right)\right.$$
$$-\frac{\partial^2}{\partial x^2}\left(\frac{\alpha^2}{r}\right)-\frac{\partial^2}{\partial y^2}\left(\frac{\beta^2}{r}\right)-\frac{\partial^2}{\partial z^2}\left(\frac{\gamma^2}{r}\right)-2\frac{\partial^2}{\partial y\partial z}\left(\frac{\beta\gamma}{r}\right)-2\frac{\partial^2}{\partial z\partial x}\left(\frac{\gamma\alpha}{r}\right)-2\frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha\beta}{r}\right)$$
$$\left.+\frac{\alpha'^2+\beta'^2+\gamma'^2}{c^2r}\right]+\frac{1}{6}\theta^3\left[\frac{\partial^3}{\partial x^3}\left(\frac{\alpha^3}{r}\right)+\frac{\partial^3}{\partial y^3}\left(\frac{\beta^3}{r}\right)+\frac{\partial^3}{\partial z^3}\left(\frac{\gamma^3}{r}\right)+3\frac{\partial^3}{\partial y^2\partial z}\left(\frac{\beta^2\gamma}{r}\right)\right.$$
$$+3\frac{\partial^3}{\partial y\,\partial z^2}\left(\frac{\beta\gamma^2}{r}\right)+3\frac{\partial^3}{\partial z^2\partial x}\left(\frac{\gamma^2\alpha}{r}\right)+3\frac{\partial^3}{\partial z\partial x^2}\left(\frac{\gamma\alpha^2}{r}\right)+3\frac{\partial^3}{\partial x^2\partial y}\left(\frac{\alpha^2\beta}{r}\right)$$
$$+3\frac{\partial^3}{\partial x\partial y^2}\left(\frac{\alpha\beta^2}{r}\right)+6\frac{\partial^3}{\partial x\partial y\partial z}\left(\frac{\alpha\beta\gamma}{r}\right)-3\frac{\partial^2}{\partial x^2}\left(\frac{\alpha\lambda}{r}\right)-3\frac{\partial^2}{\partial y^2}\left(\frac{\beta\mu}{r}\right)-3\frac{\partial^2}{\partial z^2}\left(\frac{\gamma\nu}{r}\right)$$
$$-3\frac{\partial^2}{\partial y\partial z}\left\{\frac{(\beta\nu+\gamma\mu)}{r}\right\}-3\frac{\partial^2}{\partial z\partial x}\left\{\frac{(\gamma\lambda+\alpha\nu)}{r}\right\}-3\frac{\partial^2}{\partial x\partial y}\left\{\frac{(\alpha\mu+\beta\lambda)}{r}\right\}$$
$$+\frac{\partial}{\partial x}\left(\frac{\phi}{r}\right)+\frac{\partial}{\partial y}\left(\frac{\psi}{r}\right)+\frac{\partial}{\partial z}\left(\frac{\chi}{r}\right)-3\frac{\partial}{\partial x}\left(\frac{\omega\alpha}{c^2r}\right)-3\frac{\partial}{\partial y}\left(\frac{\omega\beta}{c^2r}\right)-3\frac{\partial}{\partial z}\left(\frac{\omega\gamma}{c^2r}\right)$$
$$\left.+3\frac{\alpha'\lambda'+\beta'\mu'+\gamma'\nu'}{c^2r}\right]+\cdots \tag{90}$$

where $\omega=\alpha'^2+\beta'^2+\gamma'^2$.

Let us now calculate the rate of radiation of energy in an electromagnetic field defined by the equations

$$\left.\begin{aligned}&Q\equiv H+iE=curl\ C=-\frac{i}{c}\left[\frac{\partial C}{\partial t}+c\nabla\Gamma\right]\\&C=\frac{i}{c}\frac{\partial G}{\partial t}+i\ curl\ G,\quad \Gamma=-div\ G\\&G=\frac{g}{r}+\sum_1^3\frac{\partial}{\partial x_n}\left(\frac{g_n}{r}\right)+\sum_1^3\sum_1^3\frac{\partial^2}{\partial x_m\partial x_n}\left(\frac{g_{mn}}{r}\right)+\cdots\end{aligned}\right\} \tag{91}$$

where the vectors g, g_n, g_{mn}, etc., are functions of $t-\frac{r}{c}$ and $x_1=x, x_2=y_1$, $x_3=z$. In calculating $H+iE$ we need only terms of order $\frac{1}{r}$, consequently we need only differentiate the vector g.

Let s be a unit vector in the direction of the radius r from the origin, then $\frac{\partial}{\partial x}g_n=-\frac{1}{c}g'_n\,s_x$. When we differentiate again we need not differentiate $s_x=\frac{x}{r}$ because this would introduce a term of order $\frac{1}{r}$ and make the product of order $\frac{1}{r^2}$. We may therefore treat the components of s as constants in the subsequent differentiations.

Writing $s_x=s_1$, $s_y=s_2$, $s_z=s_3$, we may use the expression

$$G=\frac{1}{r}\left[g-\frac{1}{c}\Sigma s_n g_n'+\frac{1}{c^2}\Sigma\Sigma s_m s_n \dot{g}_{mn}'\right] \tag{92}$$

and we find that, here*

$$H+iE=-\frac{1}{c^2r}\Big[s\times g''-\frac{1}{c}(s\times\Sigma s_n g_n''')+\frac{1}{c^2}(s\times\Sigma\Sigma s_{mn}\, g_{mn}'''')$$
$$-i\Big\{s\times(s\times g'')-\frac{1}{c}s\times(s\times\Sigma s_n g_n''')+\frac{1}{c^2}s(\times s\times\Sigma\Sigma s_{mn} g_{mn}'''')\Big\}\Big] \quad (93)$$

Now if $H+iE=s\times a$ and $H-iE=s\times\bar{a}$, we have

$$2i(E\times H)=s[s\cdot(a\times\bar{a})]=s[\bar{a}\cdot(s\times\bar{a})] \quad (94)$$

Now if $a=b-i(s\times b)$ where $(s.b)=0$, we have

$$(s\times a)=(s\times b)+ib=ia \quad (95)$$

Hence in this case the radiation in the direction of s vanishes only if $(a.\bar{a})-0$, i. e., if a is a vector *zero* of magnitude; the condition $(s.b)=0$ is satisfied if

$$-c^2rb=g''-s(s\cdot g'')-\frac{1}{c}\Big\{\Sigma s_n g_n'''-s\Big(s\cdot\Sigma s_n g_n'''\Big)\Big\}+\cdots \quad (96)$$

and we have

$$H+iE=s\times a,\quad a=b-i(s\times b).$$

Hence, if the radiation in the direction of s vanishes, b must be a vector which satisfies the relation

$$b=i(s\times b) \quad (97)$$

i. e., b must be a vector in the direction of one of the isotropic lines at right angles to s. This implies that

$$s\times g''-\frac{1}{c}\Big\{s\times\Sigma s_n g_n'''\Big\{+\frac{1}{c^2}\Big\{s\times\Sigma\Sigma s_{mn} g_{mn}''''\Big\}-\cdots$$

is a vector in the direction of one of the isotropic lines just mentioned We may therefore write

$$g''-\frac{1}{c}\Sigma s_n g_n'''+\frac{1}{c^2}\Sigma\Sigma s_{mn} g_{mn}''''-\cdots$$
$$=ks+u \quad (98)$$

where u is an isotropic vector. This implies that G is of the form

$$G=curl\ S+\frac{i}{c}\frac{\partial S}{\partial t}+\nabla T \quad (99)$$

* An expression for the field strength, E, at a great distance from the origin has been published by Leigh Page, Phys. Rev., Vol. 19, (1922), p. 423.

where

$$\left.\begin{aligned} S&=\frac{w}{r}+\Sigma_n\frac{w_n}{r}+\Sigma\Sigma_{mn}\frac{w_{mn}}{r}+ \\ T&=\frac{u}{r}+\Sigma_n\frac{u_n}{r}+\Sigma\Sigma_{mn}\frac{u_{mn}}{r}+ \end{aligned}\right\} \tag{100}$$

the vectors w, w_n, w_{mn}, . . . and the scalars u, u_n, u_{mn} . . . being fuctions of $t-\frac{r}{c}$. It is clear that $\square S=0$, $\square T=0$.

When G has the above form, however, we find that $E\equiv 0$, $H\equiv 0$ and the field vanishes altogether.

The radiation can also vanish when $g''=0, g_n'''=0$, etc., but these equations imply that the electrical system consists of charges moving uniformly along straight lines and in this case the electromagnetic radiation is known to vanish. The electromagnetic radiation is known to vanish also in the case of a complete ring of electric charges in steady motion and in certain other cases [57] but these are of no interest for the purposes of the quantum theory.

Our conclusion is that the most satisfactory way of getting rid of the radiation of an electron describing an atomic orbit is to introduce the forces and radiation depending on the scalar potential Ψ.

In the case of a simple electric doublet whose moment is always parallel to OZ, we have

$$A_x=A_y=0, \quad A_z=\frac{1}{c}\frac{\partial}{\partial t}\frac{g_z}{r}, \quad \Phi=\Psi=-div\frac{g}{r}=-\frac{\partial}{\partial z}\frac{g_z}{r} \tag{101}$$

Retaining only the important terms we have

$$\left.\begin{aligned} E_x&=\frac{1}{c^2r}g_z''\frac{xz}{r^2}, & H_x&=-\frac{y}{c^2r^2}g_z'' & 2\Psi\frac{\partial^2\Psi}{\partial x\partial t}&=-\frac{2xz^2}{c^2r^5}g_z'g_z''' \\ E_y&=\frac{1}{c^2r}g_y''\frac{yz}{r^2}, & H_y&=\frac{x}{c^2r^2}g_z'' & 2\Psi\frac{\partial^2\Psi}{\partial y\partial t}&=-\frac{2yz^2}{c^2r^5}g_z'g_z''' \\ E_z&=-\frac{1}{c^2r}g_y''\frac{x^2+y^2}{r^2}, & H_z&=0 & 2\Psi\frac{\partial^2\Psi}{\partial z\partial t}&=-\frac{2z^3}{c^2r^5}g_z'g_z''' \end{aligned}\right\} \tag{102}$$

The total rate of radiation is thus

$$\frac{1}{c^3}\int_0^\pi\int_0^{2\pi} Sin\ \theta d\theta d\phi\Big[(g_z'')^2\ Sin^2\theta+2g_z'g_z'''\ Cos^2\ \theta\Big]$$

$$=\frac{8\pi}{3c^3}\Big[g_z''^2+g_z'g_z'''\Big]=\frac{8\pi}{3c^3}\frac{d}{dt}\Big[g_z'g_z''\Big] \tag{103}$$

The energy radiated in a period is thus evanescent. This result may be extended to the case in which 3 components of the moment vary.

VIII. Field Equations for a Moving Dielectric

We may pass from a medium containing isolated point charges, dipoles, quadrupoles, etc., to a medium containing a continuous volume density of electric charges, dipoles, quadrupoles, etc., by replacing the point potentials of Section VII by potentials of the type used in Section II.

Assuming that there are no magnetic poles, we write

$$C=\frac{1}{c}\frac{\partial G}{\partial t}+i\ curl\ G,\quad \Gamma=-div\ G \tag{104}$$

$$H+iE=curl\ C+curl\ A-\frac{i}{c}\frac{\partial A}{\partial t}-i\nabla\Phi \tag{105}$$

where A and Φ are defined as in Section II and G is of the form

$$G=\frac{1}{4\pi}\int\frac{dV}{r}[f]_{t-\frac{r}{c}}+\frac{1}{4\pi}\sum_{m=1}^{3}\frac{\partial}{\partial x_m}\int\frac{dV}{r}[f_m]_{t-\frac{r}{c}}$$
$$+\frac{1}{4\pi}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2}{\partial x_m\partial x_n}\int\frac{dV}{r}[f_{mn}]_{t-\frac{r}{c}}+\ .\ .\ . \tag{106}$$

where (x_1, x_2, x_3) are written for (x, y, z) respectively and the complex vectors f, f_m, f_{mn}, etc., have the forms

$$\begin{aligned} f&=g+\frac{i}{c}(v\times g) & g&=q+ip\\ f_m&=g_m+\frac{i}{c}(v\times g_m) & g_m&=q_m+ip_m\\ f_{mn}&=g_{mn}+\frac{i}{c}(v\times g_{mn}) & g_{mn}&=q_{mn}+ip_{mn}\end{aligned} \tag{107}$$

Since

$$\Box\int\frac{dV}{r}[f]_{t-\frac{r}{c}}=-4\pi f \tag{108}$$

we find on substituting these expressions in (105) that

$$\left.\begin{aligned} curl\ H&=\frac{1}{c}\left[\frac{\partial D}{\partial t}+\rho v\right], & div\ D&=\rho\\ curl\ E&=-\frac{1}{c}\frac{\partial B}{\partial t}, & div\ B&=0\end{aligned}\right\} \tag{109}$$

$$B=H+q-\frac{1}{c}(v\times p)+\sum_{m=1}^{3}\frac{\partial}{\partial x_m}\left[q_m-\frac{1}{c}(v\times p_m)\right]+$$
$$\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2}{\partial x_m\partial x_n}\left[q_{mn}-\frac{1}{c}(v\times p_{mn})\right]+\ .\ .\ . \tag{110}$$

$$D=E+p+\frac{1}{c}(v\times q)+\sum_{m=1}^{3}\frac{\partial}{\partial x_m}\left[p_m+\frac{1}{c}(v\times q_m)\right]+$$
$$\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2}{\partial x_m\partial x_n}\left[p_{mn}+\frac{1}{c}(v\times q_{mn})\right]+\ .\ .\ . \qquad (111)$$

It is thought that with suitable conventions as to the meaning of the quantities involved, these equations can be regarded as the electromagnetic equations for a material medium in which there are no conduction currents.

In writing the equations in the above form our plan has been to eliminate as far as possible all terms which might be interpreted as volume densities of electric current. Thus when B and E are treated as the fundamental vectors, the term in curl H which depends on $-\frac{1}{c}$ (v$\times p$) might be described as a Röntgen current, while according to our convention $-\frac{1}{c}(v\times p)$ is a part of the *effective magnetisation* per unit volume. This convention is similar to that recommended by Minkowski and Born.[38]

It should be noticed that Heaviside units have been used in these equations. The transition to electrostatic units is easily made. Thus the first equation would have the factor $\frac{4\pi}{c}$ instead of $\frac{1}{c}$, in the third equation all the p's and q's would be multiplied by the 4π and in the last equation E would be divided by 4π.

H. A. Lorentz obtained a set of electromagnetic equations for the description of electromagnetic phenomena in a material substance by a process of averaging,[39] in which the quantities averaged were the field vectors occurring in the fundamental equations of the theory of electrons,

$$curl\ d=-\frac{1}{c}\frac{\partial h}{\partial t}, \qquad div\ h=0 \qquad (112)$$

$$curl\ h=\frac{1}{c}\left[\frac{\partial d}{\partial t}+\rho v\right], \qquad div\ d=\rho \qquad (113)$$

which, in their turn, can be regarded as derived from the ether equations

$$\left.\begin{aligned} curl\ d_0&=-\frac{1}{c}\frac{\partial h_0}{\partial t}, \quad div\ h_0=0\\ curl\ h_0&=\frac{1}{c}\frac{\partial d_0}{\partial t}, \quad div\ d_0=0\end{aligned}\right\} \qquad (114)$$

by a process of averaging in which electric poles are considered.

Lorentz's plan was to obtain an average value of a quantity for a point P by averaging the quantity over a sphere S whose center is at P and whose radius is small in the physical sense and yet so large in comparison with the distance between electrons that S can contain an enormous number of electrons. Writing

$$\frac{1}{S}\int d\cdot dS=E,\quad \frac{1}{S}\int h\cdot dS=B \tag{115}$$

the equations (112) give

$$curl\ E=-\frac{1}{c}\frac{\partial B}{\partial t},\quad div\ B=0 \tag{116}$$

but equations (113) take a modified form depending upon the form in which we express the average values of ρ and ρv.

This process of averaging has been developed and extended by other writers in various ways and has been combined with the kinematics of the theory of relativity so as to give equations for electromagnetic phenomena in moving bodies that are compatible with that theory. It has been remarked, however, by Fokker[40] that the introduction of the ideas of relativity is unnecessary and simply complicates the analysis. The desired equations can be obtained directly by a variational process invented by Minkowski[8] and developed by Born,[38] and the invariance or the equations in the transformations of the theory of relativity can be inferred from their final form. The theory in which electrons are treated simply as point charges in the process of averaging has been worked out in a simple manner by Cunningham,[41] using both space and time averages separately, together with the kinematics of relativity. The use of an average over a space-time domain is a feature of the work of Dällenbach[42] and Fokker.[40] In some respects this method of averaging has advantages over the others, especially when one is dealing with moving matter.

The results that may be obtained by the methods of averaging appear to be no different in character from those which may be derived by the methods of Section VII and Section VIII. Fokker's variational equation (3) corresponds to our equation (69), in fact in Section VII we have really used the variational method in a slightly different form. Our analysis differs from that of Fokker and others because, for the sake of generality, we have introduced magnetic poles and doublets and have examined the detailed structure of the terms corresponding to singularities of higher order. The introduction of magnetic doublets in the terms derived from G_1, which correspond to Fokker's first variational terms, seems desirable in view of the recent ideas with regard to the magnetic properties of the electron and the experimental evidence in favour of the magneton, i. e., an electron with a magnetic moment. In the usual theory magnetization arises from the revolution of electrons round the positive nucleus of an atom and its effects appear in the second order variation, i. e., in the terms corresponding to those which arise from our function G_2. If, however, the electron itself has a magnetic moment the magnetic terms should appear at an earlier stage as in the case of terms derivable from G_1, and should differ in structure from the terms appearing in the ordinary theory. This difference in structure would, perhaps, be most easily detected in the case of crystals and some work has already been done in this direction with results that seem favourable from the point of view of the magneton theory.

The introduction of singularities of order higher than that of a dipole may seem unnecessary for the representation of the ordinary properties of a dielectric but it may be mentioned that W. Voigt[43] has used a volume distribution of quadrupoles to represent the piezoelectric properties of a crystal having a center of symmetry. The piezoelectric properties of a crystal without a center of symmetry are thought by him to be represented with sufficient accuracy by a volume distribution of ordinary dipoles. In dealing with the piezoelectric and magnetic properties of crystals it seems best* to make use of the dynamical theory of crystal lattices that has been developed by Born[44] and other writers. This theory seems also to be the best method of studying the propagation of waves of light in an isotropic or doubly refracting medium especially when attention has to be paid to the phenomena of dispersion and rotary polarisation.

IX. Constitutive Relations

The introduction of empirical constitutive relations into the theory of electromagnetism commenced with the work of Poisson[45] and may be regarded as an application, or rather an extension, of Hooke's law of the linear dependence of stress upon strain when the latter is small. In Poisson's theory of magnetism, the magnetic induction B and the magnetic force H are connected by the simple relation $B = \mu H$, where μ is the permeability of the magnetic substance. It is now known that this theory is inadequate when there are permanent magnets and strong fields,[46] for in the latter case we have the phenomenon of hysteresis.* In Weber's theory of magnetism it is supposed that a substance contains minute particles, each of which acts as a magnet, and that in the proces of magnetising a ferromagnetic substance these are turned into more or less complete alignment. Ewing[36] has recently added to this the idea that the thing that turns is neither a molecule nor an atom but something within the atom, whose orientation is to some extent controlled

* The magnetic properties of solids have also been studied recently with the aid of quantum theory. Reference may be made to a paper by P. Ehrenfest, Amsterdam Proc. Vol. 19, (1921), p. 1011, Zeitschr. f. Phys. Bd. 5, (1921), p. 35.

* For a discussion of the phenomena of paramagnetism, diamagnetism and ferromagnetism reference may be made to Ewing's book "Magnetic Induction in Iron and other bodies" and to the numerous papers of R. Gans, K. Honda, P. Weiss and other writers. See also Livens. "The Theory of Electricity." S. R. Williams, 'Science," Oct. 14, (1921), p. 339. The subject is treated very fully in the report of the committee on magnetism. Bull. Nat. Research Council Vol. 3, No. 18, 1922.

by pairs of fixed magnets, pointing in opposite directions, and is kept in a feebly stable condition when undisturbed by an external field. On the application of an external magnetic field, which is gradually increased, the orientation of a magnetic particle is at first slightly changed, the change in its moment being approximately proportional to H. Eventually, however, the particle breaks away from its feebly stable direction of equilibrium and may swing (with dissipation of energy) into a new stable state of equilibrium in which it is controlled, perhaps by two other fixed magnets pointing in opposite directions. As the external field increases still further the particle may be pulled away from this direction and be eventually set in alignment with the field.

According to this theory, then, the simple relation can be regarded as approximately correct for weak fields, μ being treated as a constant for the magnetic substance at a given temperature, when the substance is under no stress, etc. In the case of a crystal, however, it was concluded that several constants are needed to express the relation between magnetic force and magnetic induction just as several constants are needed to express the relation between stress and strain. As a result of Plücker's experiments Lord Kelvin suggested the general relations[47]

$$\left.\begin{array}{ll} B_x=\mu_{11}H_x+\mu_{12}H_y+\mu_{13}H_z & \mu_{23}=\mu_{32} \\ B_y=\mu_{21}H_x+\mu_{22}H_y+\mu_{23}H_z & \mu_{31}=\mu_{13} \\ B_z=\mu_{31}H_x+\mu_{32}H_y+\mu_{33}H_z & \mu_{12}=\mu_{21} \end{array}\right\} \quad (117)$$

Faraday's researches indicated the existence of a dielectric constant for electrical action and the simple relation $D=kE$, which was first introduced, was elaborated in Kelvin's theory of a dielectric, the final equations for a crystal being of the form*

$$\left.\begin{array}{ll} D_x=k_{11}E_x+k_{12}E_y+k_{13}E_z & k_{23}=k_{32} \\ D_y=k_{21}E_x+k_{22}E_y+k_{23}E_z & k_{31}=k_{13} \\ D_z=k_{31}E_x+k_{32}E_y+k_{33}E_z & k_{12}=k_{21} \end{array}\right\} \quad (118)$$

It will be noticed that there is a close analogy in this respect between electric and magnetic phenomena. This analogy is useful for many purposes, but it must not be pushed too far for there are some differences of a quantitative nature. Thus in the theory of a dielectric it is generally assumed that there is a separation of charges of opposite signs under the influence of an electric field, *i. e.*, the moment of an elementary electrical polarized particle may alter in magnitude as well as direction;†

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* In Maxwell's electromagnetic theory of light these equations are assumed to hold approximately also for variable fields and in particular for waves of light over a considerable portion of the long wave spectrum. For an experimental verification see H. Rubens, Zeitschr. f. Phys., Bd. I, (1920), p. 11.

† The hypothesis that the molecules are directed under the influence of an electric field was first used by Langevin in a theory of double refraction. It was afterwards used by Debye in his theory of the dependence of the dielectric constants on temperature. The subject has been discussed at some length by R. Gans, Ann. d. Phys. Bd. 64 (1921), p. 481.

while in the theory of magnetic induction it is generally assumed that the moment of a magnetically polarized particle may be changed in direction but not appreciably in magnitude.

The analogy between magnetism and electricity has been made more complete by the discovery of piezoelectricity and pyroelectricity.

A theory of these phenomena has been developed by W. Voigt[43] with the aid of empirical equations and the form of these equations has been supported by the recent work of Born[44] on the elastic properties of crystal lattices, at least as far as primary effects are concerned.

Voigt expresses the electric moment or polarization produced by given stresses or strains in the form

$$\left.\begin{aligned}
P_x &= e_{11}x_x + e_{12}y_y + e_{13}z_z + e_{14}y_z + e_{15}z_x + e_{16}x_y + \eta_{11}E_x + \eta_{12}E_y + \eta_{13}E_z,\\
P_y &= e_{21}x_x + e_{22}y_y + e_{23}z_z + e_{24}y_z + e_{25}z_x + e_{26}x_y + \eta_{21}E_x + \eta_{22}E_y + \eta_{23}E_z,\\
P_z &= e_{31}x_x + e_{32}y_y + e_{33}z_z + e_{34}y_z + e_{35}z_x + e_{36}x_y + \eta_{31}E_x + \eta_{32}E_y + \eta_{33}E_z,
\end{aligned}\right\} \quad (119)$$

where the coefficients, e_{mn}, are the *piezoelectric constants*, the coefficients η_{mn}, *polarization constants* analogous to those giving the polarization in term of the electric field when there is no strain, P_x, P_y, P_z are the components of the polarization and x_x, y_y, z_z, y_z, z_x, x_y are the six components of strain.

Voigt also expresses the general stress components in the form

$$\left.\begin{aligned}
X_x &= -c_{11}x_x - c_{12}y_y - c_{13}z_z - c_{14}y_z - c_{15}z_x - c_{16}x_y - e_{11}E_x\\
&\quad - e_{21}E_y - e_{31}E_z\\
Y_y &= -c_{21}x_x - c_{22}y_y - c_{23}z_z - c_{24}y_z - c_{25}z_x - c_{26}x_y - e_{12}E_x\\
&\quad - e_{22}E_y - e_{32}E_z\\
Z_z &= -c_{31}x_x - c_{32}y_y - c_{33}z_z - c_{34}y_z - c_{35}z_x - c_{36}x_y - e_{13}E_x\\
&\quad - e_{23}E_y - e_{33}E_z\\
Y_z &= -c_{41}x_x - c_{42}y_y - c_{43}z_z - c_{44}y_z - c_{45}z_x - c_{46}x_y - e_{14}E_x\\
&\quad - e_{24}E_y - e_{34}E_z\\
Z_x &= -c_{51}x_x - c_{52}y_y - c_{53}z_z - c_{54}y_z - c_{55}z_x - c_{56}x_y - e_{15}E_x\\
&\quad - e_{25}E_y - e_{35}E_z\\
X_y &= -c_{61}x_x - c_{62}y_y - c_{63}z_z - c_{64}y_z - c_{65}z_x - c_{66}x_y - e_{16}E_x\\
&\quad - e_{26}E_y - e_{36}E_z
\end{aligned}\right\} \quad (120)$$

where the coefficients c_{mn} are elastic constants.* Substituting the expressions for the strains derived from these equations in the preceding equations the polarization may be expressed directly in terms of the generalized stresses, and the electric force, the coefficients of the stress being the *piezoelectric moduli*.

As in the case of magnetism, these relations cannot be regarded as entirely satisfactory because the phenomena are complicated by a type

* The c's and e's are generally independent in Born's theory of a crystal, which seems to give a satisfactory foundation for the multiconstant theory of elasticity.

of hysteresis.* When there is no external electric field the electric field arising from the polarization produced by an applied stress will itself produce a secondary polarization and so we have to distinguish between primary and secondary effects; but the secondary effects are usually small; they have, however, been studied by Voigt in some special problems.[48] The pyroelectricity of a crystal is generally supposed to arise from the strains introduced by a change of temperature; but if a true pyroelectricity is eventually found to exist it may be necessary to add terms proportional to the temperature to equations (119).

According to Lord Kelvin, W. Voigt and others, a pyroelectric crystal possesses a permanent electric moment and is analogous to a permanent magnet, but under ordinary conditions the external effect of this moment is masked by a compensating effect of a surface charge of electricity which gathers on the surface. The calculations of Voigt[49] indicate that in the case of a crystal without a center of symmetry, it is not possible by means of observations to determine the magnitude of the permanent electric moment because in a deformation the effects of the geometrical and physical changes are added together in such a manner that they cannot be separated. In the case of a crystal with a center of symmetry, the theory requires modification and a separation of the geometrical and physical effects seems possible.

Piezoelectric moduli for certain acentric crystals have been calculated by Born[50] and his co-workers, using the lattice theory; the agreement with the experimental values is, however, not perfect.

In the lattice theory a certain configuration of electric charges, called the base, repeats itself periodically as we go from one cell of the lattice to another. The ordinary elastic strains arise from displacement of each base as a whole, while the piezoelectric strains which give rise to the effective electric moments arise from displacements of the charges in a base relative to each other. A state of initial strain in the base may give a permanent electric moment and it should be noticed that this moment depends on rotational terms of type

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\quad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\quad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},$$

u, v, w, being the component displacements, as well as on terms of the type

$$x_x=\frac{\partial u}{\partial x},\quad y_z=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} \tag{121}$$

which represent the components of strain. Voigt has shown, however, that observable phenomena are not influenced by these rotational terms.

* See, for instance, J. Valasek, Phys. Review, Vol. 17 (1921), p. 475; Vol. 19, (1922), p. 478.

In the case of a crystal with a center of symmetry,[49] it is apparently possible, according to Voigt, to account for the observed phenomena by means of a volume distribution of quadrupoles, the electric potential being given by

$$\Phi=\frac{\partial^2}{\partial x^2}\int\frac{\Pi_{11}}{r}dV+\frac{\partial^2}{\partial y^2}\int\frac{\Pi_{22}}{r}dV+\frac{\partial^2}{\partial z^2}\int\frac{\Pi_{33}}{r}dV+2\frac{\partial^2}{\partial y\partial z}\int\frac{\Pi_{23}}{r}dV$$
$$+2\frac{\partial^2}{\partial z\partial x}\int\frac{\Pi_{31}}{r}dV+2\frac{\partial^2}{\partial x\partial y}\int\frac{\Pi_{12}}{r}dV. \qquad (122)$$

The empirical relations adopted as a first approximation are now of the form

$$\left.\begin{aligned}\Pi_{11}&=t_1+t_{11}x_x+t_{12}y_y+t_{13}z_z+t_{14}y_z+t_{15}z_x+t_{16}x_y,\\ \Pi_{23}&=t_4+t_{41}x_x+t_{42}y_y+t_{43}z_z+t_{44}y_z+t_{45}z_x+t_{46}x_y.\end{aligned}\right\} \qquad (123)$$

where the t_n's are functions of the temperature alone and the t_{mn}'s are constants.

The phenomena of electrostriction and magnetostriction are of a different order and will not be discussed here. Reference may be made, however, to some recent work on the subject.[51]

In a conducting substance it is necessary to introduce an additional term J into the field equations of a dielectric and to postulate an empirical relation such as Ohm's law

$$J=\sigma E \qquad (124)$$

The idea of magnetic conductivity has been used by W. Arkadiew, Phys. Zeitschr. Bd. 14, (1913), p. 928, Ann. d. Phys. Bd. 65 (1921), p. 643, but is not generally adopted.

In a crystal the relation between the conduction current J and the electric force E is generally expressed to a first approximation by the following generalisation of Ohm's law.

$$\left.\begin{aligned}J_x&=\sigma_{11}E_x+\sigma_{12}E_y+\sigma_{13}E_z\\ J_y&=\sigma_{21}E_x+\sigma_{22}R_y+\sigma_{23}E_z\\ J_z&=\sigma_{31}E_x+\sigma_{32}E_y+\sigma_{33}E_z\end{aligned}\right\} \qquad (125)$$

where the coefficients σ_{mn} are constants which may be called the *conductivities* of the crystal. In a moving conductor the relation for an isotropic substance seems to be of type[52]

$$J_w-w\rho=\frac{\sigma}{\gamma}\left[E+\frac{1}{c}(w\times B)\right]_w,\ J_{\bar{w}}=\sigma\gamma\left[E+\frac{1}{c}(w\times B)\right]_{\bar{w}},\ \gamma=\frac{1}{\sqrt{1-\frac{w^2}{c^2}}} \qquad (126)$$

where the suffixes w and $\bar{w}$ are used to indicate components in the direction of w and at right angles to w respectively.

It cannot be said that a theory of this type furnishes a satisfactory theory of electric conduction. It seems better to form a definite picture of the electronic processes and the part played by the atoms and the pressure and the thermal agitation. The recent researches of Bridgman seem to provide us with a good picture and reference may be made to his papers.[53]

The use of empirical equations of the above type is recommended simply as a substitute for more precise relations depending on definite knowledge of the structure of matter and now that the properties of crystals and metals has been revealed to some extent by X-rays, attempts have been made to use the theory of lattice structures in which atoms are at the nodes of lattices and their electrons are capable of vibration. Notable success has already been obtained in this line of work and P. P. Ewald[54] has succeeded in developing a theory of the propagation of waves in a crystal in which, for the case of a long wave-length, the wave surface is a Fresnel wave-surface just as in the ordinary theory based on the empirical relations. A noteworthy feature of this theory and that of Oseen[54] is that the waves in the interior of the crystal arise from free vibrations of electric doublets at the nodes of a lattice. The conditions at the boundary offer some difficulties that are not insuperable. A precise meaning has been given to the incident wave in this theory in a recent paper by Faxén.[55]

The theory of dispersion presents some difficulties on account of the apparent incompatibility of the old theory with the quantum theory of radiation.* The old theory has, however, been developed further with some success[56] and attempts have been made to form a satisfactory theory of the natural optical activity of crystals,[57] a phenomenon that seems to be connected in some way with the piezoelectric properties.

Empirical relations can still be used to advantage in dealing with the optical properties of media when there is dispersion, if it is understood that the optical coefficients are functions of the wave-length. The coefficients may be treated as approximately constant for wave-lengths that are very different from those of the absorption bands of the substance in the case of a crystal; this is generally true in the case of long waves. The empirical relations are useful for the treatment of optical phenomena in moving bodies and in particular for a theory of the propagation of light in moving water in Fizeau's experiment and in the moving crystal in Zeeman's modified form of this experiment.[58]

Many methods of deriving constitutive relations for a moving medium have been based on the transformations of the theory of relativity. The following general method[59] seems to have advantage over the others in directness.

* The situation has been reviewed recently by P. S. Epstein, Zeitschrift für Physik, Bd. 9, 1922, p. 92. After discussing some of the attempts to partially combine quantum theory with the classical theory he concludes that a satisfactory theory of dispersion must be based on entirely new hypotheses and that, in particular, Bohr's frequency condition must be used.

We commence by writing the electromagnetic equations for a dielectric in the integral form[60]

$$\iint \pm[B_x d(y,z)+B_y d(z,x)+B_z d(x,y)+cE_x d(x,t)+cE_y d(y,t) +cE_z d(z,t)]=0 \tag{127}$$

$$\iint \pm[D_x d(y,z)+D_y d(z,x)+D_z d(x,y)-cH_x d(x,t)-cH_y d(y,t) -cH_z d(z,t)]=0 \tag{128}$$

where the integrals are taken over closed surfaces and t is expressed in terms of x, y and z by means of some arbitrary uniform law. The ambiguity in sign may be removed by choosing the sign so that the expression under the integral sign reduces to an element of an ordinary surface integral when t is constant.

The conditions to be satisfied at the boundary between two adjacent dielectrics are simply that the integrands should be continuous at the boundary, the quantities $d(y, z)$, etc., being derived from increments (dx, dy, dz, dt), $(\delta x, \delta y, \delta z, \delta t)$ which refer to points which always lie on the boundary, whether this is at rest or in motion.

The first integral form may be supposed to vanish for a moving or stationary line of magnetic induction, or for a moving surface made up of such lines, *i.e. a perfect conductor*. The second integral form may be supposed to vanish for a moving line of electric induction or for a moving surface made up of such lines, *i.e. a tube of electric induction*.

The constitutive relations should now indicate a method of expressing one integral form in terms of the other, and conversely if we can derive such a method of deduction from some rule in the absolute calculus[61] we shall have a general method of deriving a scheme of constitutive relations from the differential form or forms with which the rule is associated.[59]

There is a very simple rule associated with a general quadratic differential form

$$Q=\sum_{m,n} g_{mn}dx_m dx_n, \tag{129}$$

$$(x_1=x, \quad x_2=y, \quad x_3=z, \quad x_4=t)$$

whose coefficients are functions of x, y, z and t. The geometrical analogy is reciprocation with regard to a quadric, the integral forms corresponding to linear complexes of straight lines. The two integral forms are said to be reciprocals when their coefficients are connected by relations of type

$$\left.\begin{aligned} KB_x\sqrt{-g}&=\sum_1^4\sum_1^4 g_{2m}g_{3n}D_{mn}\\ cKE_x\sqrt{-g}&=\sum_1^4\sum_1^4 g_{1m}g_{4n}D_{mn}\end{aligned}\right\} \tag{130}$$

where

$$\left.\begin{aligned} D_{mn} &= -D_{nm} \\ D_{14} = D_x,\ D_{24} &= D_y,\ D_{34} = D_z, \\ D_{23} = -cH_x,\ D_{31} &= -cH_y,\ D_{12} = -cH_z. \end{aligned}\right\} \tag{131}$$

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{vmatrix} \tag{132}$$

and K is a constant at our disposal.

The vanishing of the quadratic form Q is supposed to indicate the way in which waves of light are propagated through the medium and to give a direct means of determining the velocity along a ray.

In the case of a stationary isotropic medium, the assumption

$$Q = dx^2 + dy^2 + dz^2 - \frac{c^2}{k\mu} dt^2 \tag{133}$$

combined with the choice $K = -\sqrt{\frac{k}{\mu}}$ gives $D = kE$, $B = \mu H$.

When we pass to the case of a uniformly moving isotropic medium by using the relativity transformation

$$x = \frac{x' + vt'}{\beta},\ y = y',\ z = z',\ t = \frac{c^2 t' + vx'}{c^2 \beta},\ c\beta = \sqrt{c^2 - v^2}, \tag{134}$$

we assume that the two integral forms and the quadratic form Q are invariants. We then have

$$Q = c^2 \frac{dx'^2 + 2v dx' dt' + v^2 dt'^2}{c^2 - v^2} + dy'^2 + dz'^2 - \frac{c^2}{k\mu} \frac{c^4 dt'^2 + 2vc^2 dx' dt' + v^2 dx'^2}{c^2(c^2 - v^2)} \tag{135}$$

With the same choice of K as before we obtain the constitutive relations

$$\left.\begin{aligned} & B_x = \mu H_x \\ & (c^2 - v^2) B_y = \left(c^2\mu - \frac{v^2}{k}\right) H_y - vc\left(\mu - \frac{1}{k}\right) D_z \\ & (c^2 - v^2 B_z = \left(c^2\mu - \frac{v^2}{k}\right) H_z + vc\left(\mu - \frac{1}{k}\right) D_y \\ & \frac{1}{k} D_x = E_x \\ & \left(\frac{c^2}{k} - \mu v^2\right) D_y + vc\left(\frac{1}{k} - \mu\right) H_z = (c^2 - v^2) E_y \\ & \left(\frac{c^2}{k} - \mu v^2\right) D_z - vc\left(\frac{1}{k} - \mu\right) H_y = (c^2 - v^2 E_z \end{aligned}\right\} \tag{136}$$

which are equivalent to those given by Einstein and Laub[62]. When $dy'=dz'=0$, the equation $Q=0$ gives for the ray velocity w

$$w=\frac{dx'}{dt'}=\frac{\pm\frac{c}{\sqrt{k\mu}}\left[1-\frac{v^2}{c^2}\right]+v\left[1-\frac{1}{k\mu}\right]}{1-\frac{v^2}{c^2k\mu}}=\pm\frac{c}{\sqrt{k\mu}}+v\left[1-\frac{1}{k\mu}\right]$$

or, if $\sqrt{k\mu}=n$

$$w=\pm\frac{c}{n}+v\left(1-\frac{1}{n^2}\right) \tag{137}$$

This is the usual equation for Fresnel's dragging coefficient.

The general equations (130) are closely connected with those adopted by Einstein[62] De Donder[63] and Wiechert[64] in their theories of gravitation. De Donder, however, adopts a slightly different definition of magnetic induction and electric force, his B and E, when multiplied by $\sqrt{-g}$ giving our B and cE. The field equations then have the form

$$\left.\begin{aligned} div\ D=\rho \quad rot\ H=C \\ div\ (B\sqrt{-g})=0,\ rot\ (E\sqrt{-g})=-\frac{\partial}{\partial t}(B\sqrt{-g}) \end{aligned}\right\} \tag{138}$$

When we adopt the conventional language of Euclidean geometry a gravitational field of Einstein's type is, in our opinion, comparable with a material medium with electric and magnetic polarizations. There is, however, the important distinction that in a gravitational field the coefficients g_{mn} satisfy certain gravitational equations which are characteristic of Einstein's theory and to obtain a material medium whose properties are associated with a quadratic differential form having coefficients of the right type we may have to introduce polarizations which travel with velocity c. Fields arising from polarization of this type will be studied in Section 10.

The above constitutive relations between the vectors B, D, E and H are not sufficiently general to cover the case of a moving crystalline medium which exhibits the phenomenon of double refraction and of course they do not give an account of the phenomena of dispersion and rotary polarization. For greater generality we may introduce, in place of the quadratic differential form, a quadratic integral form

$$I=\sum_{l,m,n,p} g_{lmnp}d(x_l,\ x_m)d(x_n,\ x_p) \tag{139}$$

$$g_{lmpn}=-g_{lmnp},\quad g_{lmnp}=-g_{mlnp} \tag{140}$$

in which the coefficients g_{lmnp} are functions of the variables x_n.

The integral forms in equations (127-128) are then assumed to be reciprocals with respect to the quadratic integral form when the relations between the vectors are of type

$$\left.\begin{aligned} \lambda B_x &= \sum_{n,p} g_{23np} D_{np} \\ c\lambda E_x &= \sum_{np} g_{14np} D_{np} \end{aligned}\right\} \tag{141}$$

where λ is at our disposal.

For the case of a stationary crystal which does not exhibit piezoelectric properties under the conditions under contemplation, we may use the quadratic integral form

$$\left.\begin{aligned} I = {} & \mu_{11}\{d(y, z)\}^2 + \mu_{22}\{d(z, x)\}^2 + \mu_{33}\{d(x, y)\}^2 + 2\mu_{23}\{d(z, x)d(z, y)\} \\ & + 2\mu_{31}d(x, y)d(y, z) + 2\mu_{12}d(y, z)d(z, x) - c^2[K_{11}\{d(x, t)\}^2 \\ & + K_{22}\{d(y, t)\}^2 + K_{33}d(z, t)^2 + 2K_{23}d(y, t)d(z, t) \\ & + 2K_{31}d(z, t)d(x, t) + 2K_{12}d(x, t)d(y, t) \end{aligned}\right\} \tag{142}$$

in which the coefficients K_{mn} are the co-factors of the constituents k_{mn} in the determinant

$$\begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix}$$

divided by the determinant itself. By means of the relativity transformation (134) this quadratic integral form is transformed into

$$\begin{aligned} I' = {} & \mu_{11}\Big\{d(y', z')\Big\}^2 + \frac{c^2\mu_{22}}{c^2-v^2}\Big\{d(z', x') + vd(z', t')\Big\}^2 \\ & + \frac{c^2\mu_{33}}{c^2-v^2}\Big\{d(x', y') - vd(y', t'\Big\}^2 \\ & + 2\frac{c^2\mu_{23}}{c^2-v^2}\Big\{d(z', x') + vd(z', t')\Big\}\Big\{d(x', y') - vd(y', t')\Big\} \\ & + 2\frac{c\mu_{31}}{\sqrt{c^2-v^2}}\Big\{d(x', y') - vd(y', t')\Big\}d(y', z') \\ & + 2\frac{c\mu_{12}}{\sqrt{c^2-v^2}}\Big\{d(z', x') + vd(z', t'\Big\}d(y', z') \\ & - c^2\Big[K_{11}\Big\{d(x', t')\Big\}^2 + \frac{K_{22}}{c^2(c^2-v^2)}\Big\{c^2d(y', t') - vd(x', y')\Big\}^2 \\ & + \frac{K_{33}}{c^2(c^2-v^2)}\Big\{c^2d(z', t') + vd(z', x')\Big\}^2 \\ & + 2\frac{K_{23}}{c^2(c^2-v^2)}\Big\{c^2d(y', t') - vd(x', y')\Big\}\Big\{c^2d(z', t') + vd(z', x')\Big\} \\ & + 2\frac{K_{31}}{c\sqrt{c^2-v^2}}d(x', t')\Big\{c^2d(z', t') + vd(z', x')\Big\} \\ & + 2\frac{K_{12}}{c\sqrt{c^2-v^2}}d(x', t')\Big\{c^2d(y', t') - vd(x', y')\Big\} \end{aligned} \tag{142}$$

and we obtain the constitutive relations for the moving crystal in the form

$$B_x' = \mu_{11}H_x' + \beta\mu_{12}\left(H_y' - \frac{v}{c}D_z'\right) + \beta\mu_{13}\left(H_z' + \frac{v}{c}D_y'\right)$$

$$B_y' = \beta^2\mu_{22}H_y' + \beta^2\mu_{23}\left(H_z' + \frac{v}{c}D_y'\right) + \beta\mu_{12}H_x' - \beta^2\frac{v}{c}K_{33}\left(D_z' + \frac{v}{c}H_y'\right)$$
$$- \beta^2\frac{v}{c}K_{23}\left(D_y' - \frac{v}{c}H_z'\right) - \beta\frac{v}{c}K_{31}D_x'$$

$$B_z' = \beta^2\mu_{33}H_z' + \beta^2\mu_{23}\left(H_y' - \frac{v}{c}D_z'\right) + \beta\mu_{31}H_x' + \beta^2\frac{v}{c}K_{22}\left(D_y' - \frac{v}{c}H_z'\right)$$
$$+ \beta^2\frac{v}{c}K_{23}\left(D_z' + \frac{v}{c}H_y'\right) + \beta\frac{v}{c}K_{12}D_x',$$

$$E_x' = K_{11}D_x' + \beta K_{12}\left(D_y' - \frac{v}{c}H_z'\right) + \beta K_{13}\left(D_z' + \frac{v}{c}H_y'\right.$$

$$E_y' = \beta^2K_{22}D_y' + \beta^2K_{23}\left(D_z' + \frac{v}{c}H_y'\right) + \beta K_{12}D_x' - \beta^2\frac{v}{c}\mu_{33}\left(H_z' + \frac{v}{c}D_y'\right)$$
$$- \beta^2\frac{v}{c}\mu_{23}\left(H_y' - \frac{v}{c}D_z'\right) - \beta\frac{v}{c}\mu_{31}H_x'$$

$$E_z' = \beta^2K_{33}D_z' + \beta^2K_{23}\left(D_y' - \frac{v}{c}H_z'\right) + \beta K_{31}D_x'$$
$$+ \beta^2\frac{v}{c}\mu_{22}\left(H_y' - \frac{v}{c}D_z'\right) + \beta^2\frac{v}{c}\left(H_z' + \frac{v}{c}D_y'\right) + \beta\frac{v}{c}\mu_{12}H_x'. \quad (143)$$

where

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and the velocity v of the crystal is in the negative direction of the axis of x.

To find the velocity, w, of light along a ray with direction cosines (l, m, n) we write

$$\frac{dx'}{l} = \frac{dy'}{m} = \frac{dz'}{n} = \frac{w\,dt'}{1} \quad (144)$$

in I' and express the condition that the resulting quadratic form in $\delta x'$, $\delta y'$, $\delta z'$, $\delta t'$ may break up into linear factors. In this way we may obtain the generalised form of Fresnel's dragging coefficient for the case of a moving crystal. An approximate formula for w has been given in an important case by H. A. Lorentz[85].

If, in particular, we write $dx' = u\,dt'$, $dy' = dz' = 0$ and neglect v^2 we obtain the determinantal equation

$$\begin{vmatrix} -c^2K_{11} & K_{12}(c^2+vw) & K_{31}(c^2+vw) \\ K_{12}(c^2+vw) & (w^2+2vw)\mu_{33}-(c^2+2vw)K_{22} & -(w^2+2vw)\mu_{23}-(c^2+2vw)K_{23} \\ K_{31}(c^2+vw) & -(w^2+2vw)\mu_{23}-(c^2+2vw)K_{23} & (w^2+2vw)\mu_{22}-(c^2+2vw)K_{33} \end{vmatrix} = 0 \qquad (145)$$

we thus find that

$$w = w_0 - v\frac{A}{B} \qquad (146)$$

where w_0 is the velocity along the ray when $v=0$ and

$$\begin{aligned} A = K_{11}[2w_0^2(\mu_{22}\mu_{33}-\mu_{23}^2) - (w_0^2+c^2)(K_{22}\mu_{22}+K_{33}\mu_{33}+2K_{23}\mu_{23}) \\ +2c^2(K_{22}K_{33}-K_{23}^2)]+2K_{12}K_{31}[(w_0^2+c^2)\mu_{23}-2c^2K_{23}] \\ +2K_{31}^2[(w_0^2+c^2)\mu_{33}-2c^2K_{22}]+2K_{12}^2[(w_0^2+c^2)\mu_{22}-2c^2K_{33}] \end{aligned} \qquad (147)$$

$$\begin{aligned} B = K_{11}[2w_0^2(\mu_{22}\mu_{33}-\mu_{23}^2) - c^2(K_{22}\mu_{22}+K_{33}\mu_{33}+2K_{23}\mu_{23})] \\ +2K_{12}K_{31}c^2\mu_{23}+K_{31}^2c^2\mu_{33}+K_{12}^2c^2\mu_{22} \end{aligned} \qquad (148)$$

When

$$K_{11}=K_{22}=K_{33}=K,\ K_{23}=K_{31}=K_{12}=0,\ \mu_{11}=\mu_{22}=\mu_{33}=\mu,$$
$$\mu_{23}=\mu_{31}=\mu_{12}=0,$$

we have

$$\frac{A}{B} = 1-\frac{K}{\mu} = 1-\frac{1}{n^2}$$

in agreement with the former result due to Fresnel. A geometrical method of deriving the ray-velocity by means of Huygens' construction has been given by A. Anderson[66].

The wave-velocity may be determined by finding the partial differential equation of the characteristics of equations (109) and (143). The wave surface is a type of Kummer's quartic surface* whose equation can be derived from that of Fresnel's surface with the aid of a Lorentz transformation.

It has been shown by Lorentz that

$$w = w_0 - \frac{v_n}{n^2} + v$$

where v_n is the component of the velocity of translation along the ray.

If the medium is a dispersive medium it has been shown by Lorentz that the ray velocity corresponding to (137) is

$$w = \frac{c}{n} \pm v\left(1-\frac{1}{n^2}\right) \mp \frac{v}{n}T\frac{dn}{dt}$$

where T is the period of vibration of the source of light.

* H. Bateman. Proc. London. Math. Soc. (2), Vol. 8, (1910), p. 375.

It has been shown by Brillouin* that in a dispersive medium energy is propagated with the group velocity so long as the frequencies of light in a plane wave are not too close to those of an absorption band of the substance. The flow of energy has also been discussed in relation to the theory of refraction at a boundary between two substances.

X. Fields With Singularities That Move Along Straight Lines With Velocity C.

There are some physical phenomena such as the photoelectric effect which seem to indicate that the energy in waves of light is concentrated around points which travel along straight lines with velocity C; and Einstein has, in his theory of the photoelectric effect†, used the hypothesis that energy travels in quanta, a quantum retaining its individuality until it is absorbed as a whole. Einstein's equation for the energy of the electrons emitted in the photo-electric effect has been confirmed by experiment; but physicists are loath to accept the theory by which it was derived, because Einstein's hypothesis seems to be incompatible with the theory of interference.

The quantum theory has, however, come to stay; and one of the outstanding problems of modern theoretical physics is to find some way of bridging the gap between the wave theory of light and the quantum theory. This problem is a difficult one, because some of the results of the two theories seem to be contradictory; thus in the wave theory of light, disturbances diverge from a source in all directions and the intensity of a disturbance diminishes as the distance from the source increases, while the quantum theory seems to require the existence of disturbances which do not spread out but are localised and retain their properties unaltered as they travel along rays. The difficulties may not be insuperable if we may assume, in accordance with the analytical result of Section IV, that there is no radiation to infinity on the whole. Energy that is emitted by one particle must then be absorbed by another; but the new electromagnetic theory depending on the function Ψ has not yet been developed sufficiently to enable one to predict the laws of such emission or absorption.

In the past, many attempts have been made to solve the radiation problem. Sir J. J. Thomson has on many occasions used the idea of discrete tubes of electric force or induction[67] and this idea has been developed by other writers[68]. H. Stanley Allen[69] has recently discussed the properties of discrete tubes of magnetic force in relation to the quantum theory and puts forward the view that magnetic tubes are

* Ann. d. Phys. Bd. 44 (1914), p. 203, Comptes Rendus. t. 173 (1921).

† See Hughes' Report. 'Bulletin of the National Research Council,' Vol. 2. April (1921).

more fundamental than electric tubes. Sir. W. H. Bragg[70], many years ago, developed an electric doublet theory of radiation which still survives in various forms at the present day. We shall have something to say about this later.

If we use a hydrodynamical analogy the emission of radiation may be compared with the production of vortices and Bohr's stationary states of electronic motion may be compared with steady stream line flow in which there is practically no eddy motion. The quantum conditions may be compared with Reynold's criterion for the commencement of turbulence, the angular momentum h being comparable with a Reynolds number LV/ν where ν is the kinematic viscosity.

Pursuing the hydrodynamical analogy, we should look for solutions of the usual field equations

$$\left.\begin{aligned} \text{curl } H &= \frac{1}{c}\frac{\partial E}{\partial t}, \quad \text{div } E = 0 \\ \text{curl } E &= -\frac{1}{c}\frac{\partial H}{\partial t}, \quad \text{div } H = 0 \end{aligned}\right\} \tag{149}$$

that have singularities different in character from those of the solutions used in the orthodox electromagnetic theory; and our plan will now be to discuss the solutions of these equations from a general point of view. If an attempt were made to solve these equations by the usual methods of the theory of differential equations, the natural plan of attack would be to look for solutions involving arbitrary functions. The first step would be to introduce the complex quantity $Q = H + iE$ so that the equations could be written in the compact form[71]

$$\text{curl } Q = -\frac{i}{c}\frac{\partial Q}{\partial t}, \quad \text{div } Q = 0 \tag{150}$$

and then to try to solve these equations by means of a vector Q of form[25]

$$Q = qf(\alpha, \beta) \tag{151}$$

where $f(\alpha, \beta)$ is an arbitrary function of two quantities α and β which are functions of x, y, z and t.

On substituting in the differential equations it is found that Q must have the two forms

$$Q = f(\alpha, \beta)(\nabla\alpha \times \nabla\beta) \tag{152}$$

$$Q = \frac{i}{c}f(\alpha, \beta)\left(\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right) \tag{153}$$

Equating these we obtain a vector equation for α and β the general solution of which may be readily obtained by taking x and y as new dependent variables. This solution may be written in the form

$$\left.\begin{aligned} z-ct&=\phi(\alpha,\beta)+(x+iy)\theta(\alpha,\beta)\\ \theta(\alpha,\beta)\,(z+ct)&=\psi(\alpha,\beta)-(x-iy)\end{aligned}\right\}\qquad(154)$$

where θ, ϕ, and ψ are arbitrary complex functions of α and β. These equations define α and β as functions of x, y, z and t. It is clear that there are many types of solution of the desired form.

In particular, if we write

$$\phi(\alpha,\beta)=\zeta(\alpha)-c\tau(\alpha)-[\xi(\alpha)+i\eta(\alpha)]\theta(\alpha,\beta)$$
$$\psi(\alpha,\beta)=\theta(\alpha,\beta)[\zeta(\alpha)+c\tau(\alpha)]+\xi(\alpha)-i\eta(\alpha)$$

the function α is defined by the equation

$$[x-\xi(\alpha)]^2+[y-\eta(\alpha)]^2+[z-\zeta(\alpha)]^2=c^2[t-\tau(\alpha)]^2$$

and β by the equation

$$\theta(\alpha,\beta)=\frac{z-\zeta(\alpha)-c[t-\tau(\alpha)]}{x-\xi(\alpha)+i[y-\eta(\alpha)]}$$

In particular, we may take $\alpha=\tau$ and $\beta=\theta$. The quantity α may then be regarded as the phase of a disturbance which originates at the moving point $[\xi(\tau),\ \eta(\tau),\ \zeta(\tau)]$; it remains constant for a point P which travels away from this moving point with velocity C along a rectilinear path. If we write

$$d\alpha=\frac{\partial\alpha}{\partial x}dx+\frac{\partial\alpha}{\partial y}dy+\frac{\partial\alpha}{\partial z}dz+\frac{\partial\alpha}{\partial t}dt$$

the coefficients of dx, dy, dz and dt, i.e. the derivatives of α remain constant as P moves. This is a consequence of the fact that α satisfies the partial differential equation

$$\left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\alpha}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial z}\right)^2=\frac{1}{c^2}\left(\frac{\partial\alpha}{\partial t}\right)^2.$$

The quantity θ which is the ratio of certain linear combinations of the derivatives of α is also a solution of this equation and is a natural type of parameter to use to specify a ray. There is, in fact, one value of θ for each ray through a point $[\xi(\tau),\ \eta(\tau),\ \zeta(\tau)]$ and vice-versa*. The quantities α and θ thus have simple physical meanings; we may call them *phase-parameter* and *ray-parameter* respectively. It is of some interest to notice that the above analysis gives us a method of deriving fields satisfying Maxwell's equations directly from these natural parameters.

* The real and imaginary parts of θ form with α a set of three independent parameters which enable the entity P to be distinguished from others as in the analysis of Section 1.

Let us consider two sets of independent inciements (dx, dy, dz, dt), $(\delta x, \delta y, \delta z, \delta t)$ of the co-ordinates of P and write

$$Y(\alpha, \theta)d(\alpha, \theta)=Q_x d(y, z)+Q_y d(z, x)+Q_z d(x, y)-ic\bar{Q}_x d(x, t)-ic\bar{Q}_y d(y, t) -ic\bar{Q}_z d(z, t)$$

then it is found from the forms of Q and $\bar{Q}$ that $\bar{Q}=Q$ and that Q satisfies equation (150), moreover the ratios of Q_x, Q_y and Q_z depend only on θ and so are constant along a ray.

The equations (150) thus occur simply as equations satisfied by certain auxiliary quantities associated with the ray parameters and it may be that this is the true meaning of Maxwell's equations. The entity that travels along a ray with velocity C may be the primary quantity and we may have to return to a kind of corpuscular theory. We may call the entity a light-corpuscle, or light-particle. But this implies that the corpuscles produce light. It is, perhaps, better to regard the entity simply as an electrical point or an aggregate of such points, using the term electrical point in the sense defined in Section I. In this sense we may speak of an electrical theory of the aether.

The field (152) specified by the functions α, β, Q may also be derived from potentials $C=A-iB$, $\Gamma=\Phi-i\Omega$, by writing

$$Q=H+iE=curl\ C=-\frac{i}{c}\left[\frac{\partial C}{\partial t}+c\nabla\Gamma\right] \tag{155}$$

where

$$C=F(\alpha, \beta)\nabla\theta,\ c\Gamma=-F(\alpha, \beta)\frac{\partial\theta}{\partial t} \tag{156}$$

$$f(\alpha, \beta)=\frac{\partial(F, \theta)}{\partial(\alpha, \beta)} \tag{157}$$

The quantity θ may be shown to be a solution of the wave equation $\Box\theta=0$ and we have the relation

$$div\ C+\frac{1}{c}\frac{\partial\Gamma}{\partial t}=0 \tag{158}$$

In terms of the functions F and θ we have

$$Q=H+iE=\nabla F\times\nabla\theta=\frac{i}{c}\left(\frac{\partial\theta}{\partial t}\nabla F-\frac{\partial F}{\partial t}\nabla\theta\right) \tag{159}$$

It should be noticed that $(Q.Q)=0$, consequently

$$E^2-H^2=0,\ (E\cdot H)=0 \tag{160}$$

We also have the relation $C^2=\Gamma^2$ which means that

$$A^2-\Phi^2=B^2-\Omega^2\quad,\quad A\cdot B=\Phi\Omega \tag{161}$$

It can be proved that the above formulae give the general solution of the problem of finding a solution of equations (149) and (160). We shall call an electromagnetic field, whose field vectors satisfy these equations, a *simple radiant field.*

If complex quantities α and β are given arbitrarily, it is not generally possible to determine real values of x, y, z and t which satisfy equations (154); but when a certain relation is satisfied, viz:

$$\phi_2+\theta_2\psi_1-\theta_1\psi_2=0, \quad (\theta=\theta_1+i\theta_2, \ \phi=\phi_1+i\phi_2, \ \psi=\psi_1+i\psi_2 \qquad (162)$$

there are ∞^1 possible values of x, y, z, t and these are successive space-time co-ordinates of a point which moves along a straight line with velocity C. We thus have a system of ∞^3 rays associated with the electromagnetic field. It can be shown that Poynting's vector at any point (x, y, z, t) is in the direction of the ray through this point, and electromagnetic energy may be supplied to flow along the rays with velocity C.

To give an example of a simple radiant field we write

$$\alpha=t-\frac{r}{c}, \ \beta=log\frac{z-r}{x+iy}, \ f(\alpha, \beta)=f(\alpha) \qquad (163)$$

where r is the distance of the point (x, y, z) from the origin and $f(\alpha)$ is a real function of α. We then have[72]

$$\left.\begin{aligned} E_x&=-\frac{xz}{r(x^2+y^2)}f\left(t-\frac{r}{c}\right) & H_x&=\frac{y}{x^2+y^2}f\left(t-\frac{r}{c}\right)\\ E_y&=-\frac{yz}{r(x^2+y^2)}f\left(t-\frac{r}{c}\right) & H_y&=-\frac{x}{x^2+y^2}f\left(t-\frac{r}{c}\right)\\ E_z&=\frac{1}{r}f\left(t-\frac{r}{c}\right) & H_z&=0 \end{aligned}\right\} \qquad (164)$$

If $f(\alpha)$ is zero except during a very short interval of time, $\delta\alpha$, the field is clearly zero except within the thin expanding spherical shell for which $\alpha=t-\dfrac{r}{c}$ has values for which $f(\alpha)\pm 0$. The field has isolated singularities which travel along the axis of z with velocity C in opposite directions, the singularities being radiated from the origin.

These singularities carry electric charges which are equal and opposite in sign as may be seen by integrating the normal electric force over the plane $z=0$. The total electric charge associated with one singularity is [25]

$$2\pi c\int_{-\infty}^{a_1} f(\alpha)\, dx=2\pi c\int_{a_1}^{a_2} f(\alpha)d\alpha$$

where $\alpha_1<\alpha<\alpha_2$ is the range of values of α for which $f(\alpha)\pm 0$.

If we write down the usual expressions for the vectors E and H in the field of an electric pole which moves with uniform velocity along a straight line we have expressions of the following type for the components at time t.

$$\left.\begin{array}{lll} E_x=(1-\gamma^2)\frac{eX}{s^3}, & E_y=(1-\gamma^2)\frac{eY}{s^3}, & E_z=(1-\gamma^2)\frac{eZ}{s^3}, \\ H_x=0 & H_y=-\gamma E_z, & H_z=\gamma E_y \end{array}\right\} \quad (165)$$

where $(x.y.z)$ are the co-ordinates of a point relative to the position of the pole at time t, $c\gamma$ is the velocity of the pole in the direction OX and

$$s=\sqrt{x^2+(1-\gamma^2)(y^2+z^2)} \quad (166)$$

Now when γ^2 approaches 1 the field tends to become evanescent except in the plane $X=0$ transverse to the direction of motion and in this plane E_y and E_z become infinite. In the field considered above, however, we had electric charges travelling with the velocity of light, and the field did not become infinite at all points of two planes perpendicular to the direction of motion.

The explanation of this paradox seems to be this: An electric charge does not *produce* a field at all *while* it travels along a straight line with velocity C. Whatever field is associated with such a charge must be supposed to have been produced when the charge was separated from a compensating charge of opposite sign or when the direction of motion of the charge was changed by a collision. The nature of this field depends, moreover, upon the manner in which the charge was separated from its fellow, i. e., the manner in which it was "created" so to speak.

In the case of the field (165) the charge must be supposed to have been created suddenly an infinitely long time ago, i.e. at time $t=-\infty$. The thin spherical shell of our former example thus becomes an infinitely thin plane shell and $f(\alpha)$ is infinite while its integral from α_1 to α_2 is finite, $\alpha_1-\alpha_2$ being infinitesimal.

In the type of field represented by the expressions (164) the charges are created more gradually, the spherical shell is not infinitely thin and this is why the electric force is not infinite all over the shell. Another peculiarity of a field of type (164) is that it may have a beginning and an end.

These properties of fields of type (152) suggest that such fields may throw some light on the nature of the aether. If the aether is supposed to be made up of electricity travelling along straight lines with velocity C, the electromagnetic fields may be produced by collisions between the aether particles. An aether particle may consist normally of an electric doublet travelling with velocity C and its constituents, according to the above idea, must be supposed to be separated gradually and not suddenly.

Simple radiant fields may be generalized either by integration or differentiation, both of these being particular cases of the general process of superposition. One method of generalisation by integration is to make the functions θ, ϕ, ψ and f functions of a parameter ω and to define a new vector $\bar{Q}$ by the equation[73]

$$\bar{Q}=\int_{\omega_1}^{\omega_2} Q d\omega \tag{166}$$

where ω_1 and ω_2 are either constants or functions of $x.y.z.t$ which satisfy equations of type

$$G_1(\alpha, \beta, \omega_1)=0, \qquad G_2(\alpha, \beta, \omega_2)=0 \tag{167}$$

G_1 and G_2 being arbitrary functions with continuous derivatives. This method of generalization is useful for the solution of diffraction problems.[74]

Another method of generalization may be described as follows:—
Let Q_s denote the vector Q when we use the function $f_s(\alpha, \beta)$ instead of $f(\alpha, \beta)$, then we may define a complex field of the first order by means of the equation

$$Q'=\sum_s \frac{\partial}{\partial x_s} Q_s, \tag{168}$$

use being now made of the notation $x_1=x$, $x_2=y$, $x_3=z$, $x_4=t$. Again, if Q_{mn} denotes the vector Q when the function $f_{mn}(\alpha, \beta)$ is used instead of $f(\alpha, \beta)$, we may define a complex field of the second order by means of the equation

$$Q''=\sum_m\sum_n \frac{\partial^2}{\partial x_m \partial x_n} Q_{mn} \tag{169}$$

and so on.

To give an example of a field of the first order let us make f constant in equations (164) and differentiate with respect to z. We obtain

$$E_x=-\frac{fx}{r^3}, \quad E_y=-\frac{fy}{r^3}, \quad E_z=-\frac{fz}{r^3}, \quad H=0. \tag{170}$$

This is the field of an electric pole at the origin. If f were not constant we should obtain the field of a pole with a varying electric charge[25] at the origin, electric charges of the same sign being projected in opposite directions from the pole at each instant. The field is in this case

$$\left.\begin{aligned}
E_x &= -\frac{x}{r^3}f\left(t-\frac{r}{c}\right)+\frac{xz^2}{cr^2(x^2+y^2)}f'\left(t-\frac{r}{c}\right),\\
H_x &= -\frac{yz}{cr(x^2+y^2)}f'\left(t-\frac{r}{c}\right)\\
E_y &= -\frac{y}{r^3}f\left(t-\frac{r}{c}\right)+\frac{yz^2}{cr^2(x^2+y^2)}f'\left(t-\frac{r}{c}\right),\\
H_y &= \frac{xz}{cr(x^2+y^2)}f'\left(t-\frac{r}{c}\right)\\
E_z &= -\frac{z}{r^3}f\left(t-\frac{r}{c}\right)-\frac{z}{cr^2}f'\left(t-\frac{r}{c}\right), \qquad H_z=0
\end{aligned}\right\} \tag{171}$$

By superposing a simple radiant field on the field just specified we can get rid of the charges projected in one of the two directions and obtain a field, which, after changing the sign of f can be written in the form

$$\left.\begin{aligned}
E_x &= \frac{x}{r^3}f\left(t-\frac{r}{c}\right)+\frac{xz}{cr^2(r+z)}f'\left(t-\frac{r}{c}\right), & H_x &= -\frac{yr}{cr^2(r+z)}f'\left(t-\frac{r}{c}\right)\\
E_y &= \frac{y}{r^3}f\left(t-\frac{r}{c}\right)+\frac{yz}{cr^2(r+z)}f'\left(t-\frac{r}{c}\right), & H_y &= \frac{xr}{cr^2(r+z)}f'\left(t-\frac{r}{c}\right)\\
E_z &= \frac{z}{r^3}f\left(t-\frac{r}{c}\right)-\frac{r-z}{cr^2}f'\left(t-\frac{r}{c}\right), & H_z &= 0.
\end{aligned}\right\} \tag{172}$$

This is the field of an electric pole whose associated electric charge varies on account of the emission of electric charges in one direction.[75]

These examples show that by means of simple radiant fields we can cancel out some or all of the singularities of fields of the first order. It can also be shown that some singularities, that move along straight lines with velocity c, can be cancelled out by superposing simple radiant fields on fields of the second order. This means, of course, that we can build up fields in which singularities moving with velocity c can be either absorbed or deflected.* These singularities may have electric or magnetic charges associated with them or they may be electric or magnetic doublets. It should be mentioned that it is possible to write down a field in which secondary singularities are projected from a primary singularity in directions which vary in an arbitrary manner with the time.[25] The primary singularity, too, may move in an arbitrary manner with a velocity which is always less than c.

* It is thought that this result may have some bearing on Bragg's theory of radiation if this is revived in some form. The author feels inclined to adopt the view that these moving singularities are directly responsible for body forces and, in particular, for the electromagnetic force per unit volume mentioned on p. 101. If this view be adopted, it may be advantageous to change the signs of electromagnetic energy and momentum and limit only *material* energy to be positive.

By suitably choosing the law of variation of projection of the secondary singularities, it is possible to make the locus of the secondary singularities a line of electric force of a moving electric pole.[20] If, moreover, we superpose a number of fields of this type it is possible, at least in some cases, to make the field zero outside a number of discrete tubes of electric force of a moving electric pole. In this way it would be possible to derive a theory of discrete Faraday tubes from Maxwell's equations but it is doubtful whether the theory would be of any physical interest.

The fact that a secondary singularity retains its properties unaltered (in some of the simplest fields at least) while it moves with velocity c along a straight line, suggests that it may be possible to base a satisfactory theory of radiation on Maxwell's equations but we are quite unable to say at present what form this theory should take. There are many difficulties.

If, for instance, we suppose that the secondary singularities carry electric charges and we regard them as symbols for discrete electric charges of small but finite size we must account for the generation of these charges. The first possibility that occurs to one is that the charge of an electron may not be always the same but may depend on the magnitude of the function Ψ. This seems reasonable because it appears from the analysis of Section III that an electron cannot move continuously without loss of charge from a region where the potential Ψ has a value Ψ_0, which is practically constant, to another region where the potential Ψ_0 has another value Ψ_0 which is practically constant. There are in fact four conditions to be satisfied in order that the continuous motion may be possible and there are only three available constants.

Another possibility is that the emitted charges may be derived from electric doublets already existing in the æther and travelling with velocity c along straight lines. An apparent difficulty occurs here because work is needed to separate the component charges of a doublet. It should be pointed out, however, that if the electric field of an electric pole is not exactly the same in all directions—and this is true when the pole has an accelerated motion—it is possible for the work done on one radiated constituent of a doublet against the field of the pole to be more than balanced by the energy gained in separating the other constituent of the doublet from the pole.

It is interesting to inquire whether the radiation from a moving electric pole can be cancelled out in practically all directions by superposing on it fields of type (152) in which secondary singularities are fired out from the moving pole with velocity c.

We have seen that the electric and magnetic vectors in the field of the moving pole can be written in the form

$$H_x = \frac{\partial(\sigma, \tau)}{\partial(y, z)}, \quad E_x = \frac{1}{c}\frac{\partial(\sigma, \tau)}{\partial(x, t)}, \tag{173}$$

where σ and τ are defined by the equations (33), (35) and (36). The radiation depends only on the terms of order $\frac{1}{r}$ and these may be written in the form

$$H_x^* = \frac{\partial(\sigma^*, \tau)}{\partial(y, z)}, \quad cE_x^* = \frac{\partial(\sigma^*, \tau)}{\partial(x, t)} \tag{174}$$

where σ^* is what σ becomes when we omit the term $c^2 - v^2$ in K. To get rid of the radiation we must completely cancel out these terms of order $\frac{1}{r}$. Now a simple radiant field in which secondary singularities are radiated from the moving pole is given by formulae of type

$$H_x + iE_x = \frac{\partial(\lambda, \tau)}{\partial(y, z)} = \frac{i}{c}\frac{\partial(\lambda, \tau)}{\partial(x, t)} \tag{175}$$

where $\lambda = G(\beta, \tau)$ and

$$\beta = \frac{z - \zeta(\tau) - c(t-\tau)}{x - \xi(\tau) + i[y - \eta(\tau)]}. \tag{176}$$

The terms in E and H are now all of order $\frac{1}{r}$ but they will not generally cancel E^* and H^* except, perhaps, when the secondary singularities form a continuous tube or fill the whole of space or a part of it. If E^* and H^* can be cancelled out in any region of space the resultant field will be of type [25]

$$H = \nabla\omega \times \nabla\tau, \quad E = \frac{1}{c}\left[\frac{\partial\tau}{\partial t}\Delta\omega - \frac{\partial\omega}{\partial t}\Delta\tau\right] \tag{177}$$

where

$$\omega = e\frac{c^2 - \xi'^2 - \eta'^2 - \zeta'^2}{(x-\xi)\xi' + (y-\eta)\eta' + (z-\zeta)\zeta' - c^2(t-\tau)} \tag{178}$$

This field reduces to an electrostatic field of the ordinary type when the primary singularity is stationary.

There is a possibility that the terms E^* and H^* may be cancelled out by superposing on the field of the electric pole not only a combination of simple radiant fields, having the pole as primary singularity, but also a combination of fields of the first, second and higher orders with the pole as primary singularity. It is possible, indeed, to cancel out the field of the pole altogether and there is a danger that when we cancel out the radiation terms this is just what will happen. There will, however, be a generalization of (177) which reduces to a combination of the field of an electric pole, dipole, quadrupole, etc., when the primary singularity is stationary; the field reducing to a dipole's field when $v=0$ may be obtained from (177) by differentiating with regard to x, y, z, multiplying the results by arbitrary constants and adding.

In a field of type (177) there are no terms of order $\frac{1}{r}$ in the expressions for E and H, consequently there is no radiation of energy to infinity as the flux represented by Poynting's vector is concerned. There is, however, a flow of electricity to infinity in the case of an accelerated motion of the primary singularity, electricity being projected from this moving point in all directions and travelling along straight lines with the velocity of light. The electric charge associated with the pole, however, remains constant and so the emitted positive electricity must be just balanced by the emitted negative electricity. The field may thus be regarded as being produced by the breaking up of electric doublets at the primary singularity and since there is no radiation of energy to infinity on the whole as represented by the Poynting flux the energy furnished by the repulsions between charges of like sign must be sufficient to balance the work done in separating the charges of opposite signs. This may not, however, be right because it may be necessary to associate mass and energy of non electromagnetic type with the electricity that flows to infinity and then this type of field would not furnish a case of no radiation. The question as to whether it is necessary to associate electromagnetic mass or a mass of other type with an electric charge travelling with velocity c is a difficult one to answer; so much depends on the way in which the charge is generated by a process of electric separation. It is usual to say that an electron would have an infinite mass if it could travel along a straight line with the velocity of light, but this may be because the field of the electron is regarded as the limiting case of the field obtained by superposing fields of point charges of Lienard's type and we have seen that in such fields the field vector E becomes infinite all over a plane when the velocity is c and the motion is rectilinear. In a simple radiant field, on the other hand, we have electric charges moving with velocity c and the field vectors are not infinite except at the electric charges. If, then, we superpose simple radiant fields so as to obtain a volume distribution of such electric charges moving with velocity c, it may not be necessary to associate an *infinite* mass with these charges, but it may still be necessary to associate a finite mass with them. This is a point on which the author feels undecided. If a finite mass must be associated with these electric charges we have a radiation of matter to infinity in a field of type (177) and this is just as bad as a radiation of energy given by the Poynting flux.

It appears then that the solution of the problem of no radiation suggested in Section IV is preferable to the one given here. The field (177) is a particular case of a more general field specified by the equation[76]

$$H+iE=\nabla\theta\times\nabla\tau+\frac{i}{c}\left[\frac{\partial\tau}{\partial t}\nabla\theta-\frac{\partial\theta}{\partial t}\nabla\tau\right]. \tag{179}$$

where

$$\theta=\frac{\alpha(x-\xi)+\beta(y-\eta)+\gamma(z-\zeta)+\delta}{(x-\xi)\xi'+(y-\eta)\eta'+(z-\zeta)\zeta'-c^2(t-\tau)} \tag{180}$$

and α, β, γ and δ are arbitrary complex functions of τ. When δ is real, the lines of electric force in such a field may be obtained by solving a Riccatian equation. When δ is imaginary, the lines of magnetic force may be obtained in a similar manner.

Fields of this type have been studied by the author and by Leigh Page.[77] The latter has used them as a basis of a theory of radiation and has invented a new type of electron whose surface is the locus of the primary singularities of fields of this type, the fields being of such a nature that the electric vectors cancel out in the interior of a stationary electron, while the magnetic vectors cancel out in the external region leaving an electrostatic field of the ordinary type. Page calls his primary fields rotating electric fields and finds that in a complete rotation the energy radiated is almost exactly $\frac{1}{32}$ hν, where ν is the frequency of rotation and h is Planck's constant. In an electromagnetic field of Page's type, magnetism is radiated from the primary singularity and travels along straight lines with velocity c.

Many physicists are of the opinion that Planck's constant h is connected in some way with magnetism. In his work on the magneton or ring-electron[78] S. B. McLaren found that the angular momentum is proportional to the number of tubes of magnetic induction linked through the ring's aperture and to the number of tubes of electric induction terminating on its surface. This result has been recently confirmed and developed by H. Stanley Allen[80] who points out some interesting connections with quantum theory and extends the result to the case in which any number of point charges are revolving round an axis with a common angular velocity. Allen shows that when quantum theory indicates that the mean value of the kinetic energy of an electron describing a periodic orbit with frequency $\frac{1}{\nu}$ is equal to $\frac{1}{2}nh\nu$, this may be regarded also as $\frac{1}{2}Ne\nu$ where $N=n\frac{h}{e}$ is the number of magnetic tubes associated with the moving charge and passing through its orbit. This is an extension of a result given by A. L. Bernoulli.

A connection between quantum theory and magnetism is also suggested by the hydrodynamical analogy, for magnetism is to some extent analogous to eddy motion in an electrical aether.

In the solution by Helmholtz[79] and Lord Kelvin[80] of d'Alembert's hydrodynamical paradox of the absence of resistance in the continuous potential flow of a fluid round an obstacle, it was pointed out that if the pressure p vanishes at any point, then the continuous flow must cease and vortices or some type of discontinuity must form. It is interesting to inquire whether there is any function analogous to the pressure in the case of an electromagnetic field.

Let us write

$$c[E\times H]=qW=\tfrac{1}{2}q(E^2+H^2)$$

and regard q for the moment as analogous to a velocity. This analogy must not be carried too far, for, as Dr. Epstein has remarked to me, q does not generally* transform like a velocity in the transformations of the theory of relativity. We have the identity

$$W^2(c^2-q^2)=\frac{c^2}{4}\left[(E^2-H^2)^2+4(E\cdot H)^2\right]=c^2P^2 \tag{181}$$

where P is Cunningham's principal stress.†

Writing $W=\rho c^2$, where ρ is a kind of mass density, we have an equation

$$\frac{\partial\rho}{\partial t}+div\ (\rho q)=0 \tag{182}$$

analogous to the hydrodynamical equation of continuity and if we neglect squares and higher powers of q^2/c^2 we obtain from (181) an equation

$$\frac{P}{\rho}+\frac{1}{2}q^2=constant \tag{183}$$

analogous to Bernoulli's integral. The quantity P is thus in some respects analogous to the pressure in hydrodynamics.[81]

P is always greater than zero in the case of the field of a moving electric pole but in the case of the field of two electric poles this is not always the case. Let us consider for instance the case of a Hertzian dipole. The field vectors in polar co-ordinates are[82]

$$\left.\begin{aligned}E_r&=\frac{2\cos\theta}{r^2}\left\{f'(ct-r)+\frac{1}{r}f(ct-r)\right\}\\E\theta&=\frac{\sin\theta}{r}\left\{f''(ct-r)+\frac{1}{r}f'(ct-r)+\frac{1}{r^2}f(ct-r)\right\}\\H\phi&=\frac{\sin\theta}{r}\left\{f''(ct-r)+\frac{1}{r}f'(ct-r)\right\}\end{aligned}\right\} \tag{184}$$

* It does transform like a velocity in the case when $q^2=c^2$.

† E. Cunningham, "The Principle of Relativity," Ch. XV.

When $\theta = \frac{\pi}{2}$ we have $(E \cdot H) = 0$ and $E^2 - H^2 = 0$ whenever $f(ct-r) = 0$. There are thus places where $P = 0$.

It seems likely also that P will vanish in the case of Bohr's hydrogen atom when the charges are treated as point charges, for $(E \cdot H) = 0$ in the plane of the orbit and there are points at which the electric vectors due to the two charges nearly cancel and at such points H may be as big as E and may make $E^2 - H^2 = 0$.

An attempt to develop a hydrodynamical theory of the electromagnetic field has been made by A. Korn.[33] He uses the idea of pulsating spheres in a practically incompressible fluid and adopts a modification of the Principle of d'Alembert. Korn's equations for the aether are of type

$$\left.\begin{aligned} curl\ v_2 &= \frac{1}{c}\left[\frac{\partial v_1}{\partial t} + (v_0 \cdot \nabla)v_1 - (v_1 \cdot \nabla)v_0\right], \quad div\ v_1 = 0 \\ curl\ v_1 &= -\frac{1}{c}\left[\frac{\partial v_2}{\partial t} + (v_0 \cdot \nabla)v_2 - (v_2 \cdot \nabla)v_0\right], \quad div\ v_2 = 0 \end{aligned}\right\} \tag{185}$$

where the vector v_0 may be interpreted as the *visible* velocity of the fluid and v_1, v_2 as coefficients which occur in the expression of the actual instantaneous velocity in the form

$$v = v_0(x, y, z, t) + v_1(x, y, z, t)\ Cos\frac{2\pi t}{T} + v_2(x, y, z, t)\ Sin\frac{2\pi t}{T}. \tag{186}$$

Korn's theory has been developed in detail but has not yet thrown any light on the difficulties presented by the quantum theory.* It should be noticed that the above equations for the aether are slightly different from the usual Maxwell equations.

J. H. Jeans[34] has expressed the opinion that the equations for the aether may really need modification and that the exact equations may involve the fundamental constants e and h of experimental physics.

Modified forms of the equations for the aether actually occur in the theories of gravitation of Einstein, Wiechert and others. According to Reissner,[35] however, the field of an electric pole in the Einstein theory differs only slightly from the field of an electric pole in the ordinary theory. Something more radical seems to be needed to account for the peculiarities of the quantum theory of radiation. Everything seems to point to the conclusion that it is the stress-energy tensor which needs modification and this involves a modification of the force equation.

Meg Nad Saha[32] has proposed a new law of electromagnetic force

* In two recent notes. Phys. Zeitschr. Bd. 20 (1919), p. 491, Bd. 21 (1920), p. 97. Korn does, however, propose a theory of line spectra based on his idea of pulsating spheres and indicates a possible connection with quantum theory.

which is regarded favorably by A. C. Crehore,[36] who has used it in atomic theory. Another law of force has already been suggested in Section II.

From the point of view of Einstein's theory of gravitation, all the foregoing analysis is simply an approximation and light does not travel along straight lines with velocity c in a gravitational field if we still use the language of Euclidean geometry.

Einstein's theory indicates that greater force of expression may be attained by using the language of non-Euclidean geometry. It is, however, still advantageous for some purposes to use the familiar language of Euclidean geometry and then the question arises as to how a gravitational field should be described in terms of familiar entities.

In Einstein's theory, if suitable interpretations are given to the vectors, the electromagnetic equations for the aether are of type

$$\left.\begin{aligned} curl\, H &= \frac{1}{c}\frac{\partial D}{\partial t}, \quad div\, D=0 \\ curl\, E &= -\frac{1}{c}\frac{\partial B}{\partial t}, \quad div\, B=0 \end{aligned}\right\} \tag{187}$$

where D, H, B and E are connected by equations of type (138) in which the coefficients g_{mn} are Einstein's gravitational potentials defined by means of an invariant quadratic form Q. Now if we use the ordinary language of Euclidean geometry these equations imply the existence of electric and magnetic polarizations in the aether and there seems no reason why an Einsteinian electromagnetic field should not be built up by superposition from fields which satisfy Maxwell's equations. There may, however, be this difference between Einstein's aether and an ordinary ponderable medium. Whereas in the case of a ponderable medium we need use only fields in which electric charges move with velocities numerically less than c, to obtain a field in Einstein's aether it may be necessary to introduce fields with singularities which move with velocity c. It seems likely then that electromagnetic fields in Einstein's aether can be built up from simple radiant fields and fields of the first, second and higher orders. In this connection it may be noticed that we may already have a tensor of the first rank associated with a field of the first order and a tensor of the second rank associated with a field of the second order, viz. the tensors defined by the functions f_m, f_{mn}. The definition of a gravitational tensor in terms of electrical quantities does not then seem impossible; indeed it is possible that gravitation may be connected in some way with a variation of the electronic or nuclear electric charge.

In deriving electromagnetic equations of Einstein's type from fields satisfying the usual electronic equations we can eliminate the currents and space charge in the æther by introducing a volume polarization and abandoning the simple relations $D=E$, $B=H$. To illustrate this let us consider the field

$$B=\nabla\sigma_0\times\nabla\tau \quad , \quad E=\frac{1}{c}\left[\frac{\partial\tau}{\partial t}\nabla\sigma_0-\frac{\partial\sigma_0}{\partial t}\nabla\tau\right] \tag{188}$$

where

$$\sigma_0=\frac{(x-\xi)\alpha+(y-\eta)\beta+(z-\zeta)\gamma+c^2-v^2}{M} \tag{189}$$

the notation being that of Section 4, α, β and γ being arbitrary real functions of τ.

We have in this case

$$div\ E=-\frac{k}{c}\frac{\partial\tau}{\partial t}, \qquad curl\ B-\frac{1}{c}\frac{\partial E}{\partial t}=k\nabla\tau \tag{190}$$

where k is a complicated function of x, y, z and t whose exact form does not at present interest us. If now we write

$$H=\nabla\sigma_1\times\nabla\tau, \qquad D=\frac{1}{c}\left[\frac{\partial\tau}{\partial t}\nabla\sigma_1-\frac{\partial\sigma_1}{\partial t}\nabla\tau\right] \tag{191}$$

where

$$\sigma_1=\frac{(x-\xi)\xi''+(y-\eta)\eta''+(z-\zeta)\zeta''+c^2-v^2}{M} \tag{192}$$

we have

$$curl\ H=\frac{1}{c}\frac{\partial D}{\partial t}, \qquad div\ D=0 \tag{193}$$

$$\left.\begin{array}{lll} B=H+q, & D=E+p, & q=-(s\times p) \\ p=\frac{1}{c}\left[\frac{\partial\tau}{\partial t}\nabla\sigma_2-\frac{\partial\sigma_2}{\partial t}\nabla\tau\right], & & \sigma_2=\sigma_1-\sigma_0 \\ q=-\nabla\tau\times\nabla\sigma_2, & & \end{array}\right\} \tag{194}$$

s being a unit vector having the same direction as $-\nabla\tau$. It is easy to see that the polarization p is everywhere at right angles to the ray, i. e. to s, and must be regarded as travelling with the velocity of light. The preceding equations may, in fact, be compared with the corresponding equations for a moving dielectric.

In conclusion it may be remarked that a field differing only slightly from a field satisfying Maxwell's equations for the aether may be obtained by averaging the field vectors for any given set of moving charges taking c as a quantity which varies within a narrow range from c_1 to c_2. In such a field there may be no radiation of energy to infinity.

XI. Dynamical Equations of Motion

It is generally believed that electrodynamics can be based on the well known principles of classical mechanics but mathematicians reserve the privilege of being allowed to alter these if necessary to make electrodynamics agree with the quantum theory of radiation.

The general dynamical argument in favour of a dynamical theory of the electromagnetic field was first formulated by Maxwell[87] for the case of the field surrounding a system of linear conductors. His analysis was subsequently extended to cover the more general case, firstly by von Helmholtz[88] and Lorentz,[89] later by Larmor,[90] Macdonald,[91] Abraham[3] and others.[42] In most of the investigations use is made of a Principle of Least Action in the simple form

$$\delta \int_{t_1}^{t_2} L dt \equiv \delta \int_{t_1}^{t_2} (T-W) dt = 0 \tag{195}$$

wherein T denotes the kinetic energy and W the potential energy of the system in any configuration. The principle is supposed to be formulated in terms of any co-ordinates that are sufficient to specify the configuration in accordance with its known properties and connexions and the variation is supposed to refer to a fixed time of passage of the system from the initial to the final configuration. In the more recent investigations of Livens[92] it is recognized that this simple form of Hamilton's principle involves a complete knowledge of the constitution of the system and that before it can strictly be applied it is necessary to know the exact values of the kinetic and potential energies expressed properly in terms of co-ordinates and velocities. As, however, we have to deal with a system whose ultimate constitution is wholly or partly unknown it is necessary to use a modified form of the principle allowing for a possible ignorance of the constitution of the system.

Suppose that the Lagrangian function is expressible in terms of a certain number of variables $x_1, x_2 \cdot \cdot \cdot x_k$, which are known to be connected with the chosen co-ordinates q and their velocities q by a series of relations of type

$$M_s = 0 \tag{196}$$

M_s being a function of the co-ordinates q, the velocities $\dot{q}$, the variables x and the differential coefficients of these latter variables with respect to the time. The usual method of procedure is to introduce a set of multipliers λ_s, functions of the time, and then to consider the variations of the integral

$$\int_{t_1}^{t_2} (L + \Sigma \lambda_s M_s) dt \tag{197}$$

where the q's and x's undergo independent variations in the case when only the first derivatives appear in our expression under the integral sign.

The variational principle then leads to an equation of type

$$\frac{d}{dt}\left(\Sigma\lambda_s\frac{\partial M_s}{\partial \dot{x}}\right)-\Sigma\lambda_s\frac{\partial M_s}{\partial x}-\frac{\partial L}{\partial x} \tag{198}$$

for each variable x and an equation of type

$$\frac{d}{dt}\left(\Sigma\lambda_s\frac{\partial M_s}{\partial \dot{q}}\right)-\Sigma\lambda_s\frac{\partial M_s}{\partial q}=0 \tag{199}$$

for each q co-ordinate.

By applying this method Livens has given a very complete and general presentation of the dynamical argument when gravitational actions are not considered in detail as in Einstein's theory. His results are quite compatible with equations of the type considered in Section VIII. An interesting feature of the method is that the electromagnetic potentials are introduced as coefficients of type λ.

The most important result of this type of dynamical reasoning is that the force F exerted by an electromagnetic field on an electric charge, e, moving with velocity v can be represented with considerable accuracy by the vector

$$e\left[E+\frac{1}{c}(v\times B)\right]=F, \tag{200}$$

a correction being necessary, perhaps, if there is radiation of energy, to allow for the reaction of the radiation. A correction may be necessary, too, when the charge is surrounded by a distribution of electric doublets, as was pointed out by Lorentz[93] in his theory of refraction and dispersion.

The Lagrangian function has been calculated for some systems of electric charges and some interesting results have been obtained.[94]

A dynamical investigation of a somewhat different character that is worthy of notice is that of E. Guillaume,[95] who derives the electromagnetic equations with the aid of Appell's general dynamical equations.*

The variational method was extended by G. Mie[9] in his theory of matter and has been used with great success by Einstein,[96] De Donder,[63] Lorentz[96] and Hilbert[97] in Einstein's theory of gravitation. It has also been used by Weyl[98] in his extension of Einstein's theory.

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* These equations are applicable to non-holonomic systems. The peculiarities of the quantum theory of radiation suggest that an electron describing a non-radiating orbit is in some respect analogous to a rolling sphere and an electron which is radiating energy to a sliding sphere. But the analogy with a ball that sometimes rolls and sometimes slides is not of much use.

De Donder expresses the variational principle in the general form

$$\delta\int(a+bC+\Lambda)\sqrt{-g}\,dx_1dx_2dx_3dx_4=o \tag{201}$$

where a and b are universal constants, g is defined by equation (130). C is given by

$$C=\sum_{\alpha}\sum_{\beta}g^{\alpha\beta}G_{\alpha\beta} \tag{202}$$

$$G_{\alpha\beta}=\sum_{\sigma}\sum_{\tau}\left[\frac{\partial}{\partial x_\beta}\begin{Bmatrix}\alpha\sigma\\ \sigma\end{Bmatrix}-\frac{\partial}{\partial x_\sigma}\begin{Bmatrix}\alpha\beta\\ \sigma\end{Bmatrix}+\begin{Bmatrix}\beta\tau\\ \sigma\end{Bmatrix}\begin{Bmatrix}\alpha\sigma\\ \tau\end{Bmatrix}-\begin{Bmatrix}\sigma\tau\\ \sigma\end{Bmatrix}\begin{Bmatrix}\alpha\beta\\ \tau\end{Bmatrix}\right] \tag{203}$$

where

$$\begin{Bmatrix}\alpha\beta\\ \gamma\end{Bmatrix}=\Sigma g^{\gamma\nu}\begin{bmatrix}\alpha\beta\\ \nu\end{bmatrix} \tag{204}$$

$$\begin{bmatrix}\alpha\beta\\ \nu\end{bmatrix}=\frac{1}{2}\left(g_{\alpha\nu,\,\beta}+g_{\beta\nu,\,\alpha}-g_{\alpha\beta,\,\nu}\right) \tag{205}$$

$$g_{\alpha\beta,\,i}=\frac{\partial g_{\alpha\beta}}{\partial x_i} \tag{206}$$

and $g^{\alpha\beta}$ is the minor of the constituent $g_{\alpha\beta}$ in g divided by g. The quantity C is called the total curvature of the space time. The quantity Λ depends on the gravitational potentials, and their derivatives and on the quantities which define the electromagnetic field and the distribution of matter. Various expressions for Λ which are generalisations of $\frac{1}{2}(B^2-E^2)$ have been suggested but there seems to be some uncertainty as to the exact form.

The calculus of variations leads to the equations

$$2bG_{\alpha\beta}\sqrt{-g}-\lambda g_{\alpha\beta}=C_{\alpha\beta} \tag{207}$$

where $C_{\alpha\beta}$ are the components of the symmetrical tensor of the electromagnetic and mass fields. These are Einstein's gravitational equations.

Many attempts have been made to derive the structure of the electron from Einstein's theory and its generalizations but they do not seem, in the writer's opinion, to be as promising as that sketched in Section III.

In connection with the problem of deriving dynamical equations of motion of the usual type from the analysis of Section II it may be recalled that Lorentz,[4] Abraham,[3] Born[99] and others* have derived the dynamical equations of motion of an electron by equating to zero the integral

$$\int\rho\left[E+\frac{1}{c}(v\times H)\right]dxdydz \tag{208}$$

* H. Poincaré, Archives Néerlandaises, t. 5 (1900), p. 252, J. Larmor, l.c. E. Kohl, Ann. d. Phys. Bd. 13 (1904), p. 770.

taken over the electron.* This is in a sense a generalization of the Principle of d'Alembert. The equations of motion derived from this principle may be expressed in the approximate form†

$$mf - n\dot{f} = e\left\{E_0 + \frac{1}{c}(v \times H_0)\right\}$$

where f is the acceleration of the electron, e, the total charge $(E_0 H_0)$ the electromagnetic field arising from external charges, a the radius of the electron and

$$m = \frac{e^2}{6\pi a c^2}, \qquad n = \frac{e^2}{6\pi c^3}.$$

In the case of uniform circular motion the term $-n\dot{f}$ represents the drag due to the electromagnetic radiation.

A natural generalization of this result may perhaps, be obtained by equating to zero the integral

$$\int\left\{\rho\left[E + \frac{1}{c}(v \times H) + 2\underline{\Psi}\nabla\left(\rho\sqrt{1 - \frac{v^2}{c^2}}\right)\right\}dxdydz$$

taken over the electron. The exact form of the equations of motion has not yet been obtained.

In conclusion it may be remarked that an extension of Section II is needed which will take into account the action of a gravitational field. The generalization of $\rho\sqrt{1 - \frac{v^2}{c^2}}$ can be written down immediately. in the form

$$\rho g^{-\frac{1}{2}}\left[\Sigma g_{\alpha\beta}\frac{dx_\alpha}{dt}\frac{dy_\alpha}{dt}\right]^{\frac{1}{2}}$$

but the writer feels uncertain as to the exact form which should be given to the stress-energy tensor.§

* The principle used by Born is slightly different from this. He reduces the electron to a state in which it is momentarily at rest by using a relativity transformation and integrates the product of the stationary ρ and the stationary force over the stationary form of the electron. This procedure is suggested by the theory of relativity. Another form of the principle is used by E. Fermi, Phys. Zeitschr. (1922), p. 340.

† The analysis is given in full by Leigh Page "An Introduction to Electrodynamics," p. 50.

§ It should be mentioned that another tensor has been found which also leads to the equations (13 and has some advantages over the tensor defined by equations (11). With both tensors there is, however, the drawback that both positive and negative energy are radiated outwards from an accelerated electric pole.

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Part IV
THE TROUTON-NOBLE EXPERIMENT

By E. H. Kennard

Symbols

c = velocity of light.
E = electric intensity in electrostatic units.
f = force per unit charge (stationary or moving).
G = electromagnetic momentum, g = same per unit volume.
H = magnetic intensity in electromagnetic units.
K = dielectric constant.
L = torque.
N = number of doublets per unit volume.
P = polarization (displacement of electricity in dielectric matter).
U = electric energy in ether = $\iiint E^2/8\pi \, d\tau$.
u = velocity of convection.
V = electrostatic potential.
v = velocity of motion of electricity.
$\beta = u/c$.
$\gamma = 1/\sqrt{1-\beta^2}$.
θ = angle of rotation of condenser.
ρ = volume density of electricity in electrostatic units.
σ = surface density of electricity in electrostatic units.
φ = retarded scalar potential.
Ψ = convection potential.
$\cdot \times$ denotes scalar and vector products, resp.
∇ denotes gradient.
Vectors are in black-face type.

I. Genesis of the Experiment

1. *The Trouton-Noble experiment* originated in a suggestion of G. F. FitzGerald's. We quote F. T. Trouton (1902)[1]: "The fundamental idea of the experiment is that a charged condenser, when moving through the ether, with its plates edgeways to the direction of motion, possesses a magnetic field between the plates in consequence of its motion" · · · "The question then naturally arises as to the source supplying the energy required to produce this magnetic field." "FitzGerald's view" · · · "was that it would be found to be supplied through there being a mechanical drag on the condenser itself at the moment of charging, very similar to that which would occur were the mass of any body situated on the surface of the Earth to suddenly become greater."

An impulse due to such a drag was looked for by Trouton, using a condenser mounted on one end of a cross arm, but with a negative

[1] Scientific writings of G. F. FitzGerald, p. 557.

result; the experiment is described in the paper just cited. Unfortunately the method employed to test the sensitiveness of the apparatus was dynamically fallacious and the effect was really far too small to be detected[2] (see also below, Sec. 5).

Having thus convinced himself that the FitzGerald drag did not exist, Trouton came to the conclusion that a charged condenser moving through the ether with its plates inclined to the direction of motion ought to experience a torque tending to turn it so as to diminish its magnetic energy and therefore so as to set its plates perpendicular to the motion. Modern theory confirms this conclusion except that the direction of the torque is just the opposite of that described.

This effect was sought in the Trouton-Noble experiment.

II. The Experiment

2. *Description.*[1] A mica-plate condenser 7.7 cm. in diameter and having a capacity of 0.0037 microfarads was suspended with its plates vertical from a phosphor-bronze strip 37 cm. long. The condenser was charged to 2100 volts by way of the suspension and a separate wire on the bottom dipping into dilute sulphuric acid. Deflections about the vertical axis were observed upon charging the condenser, by means of a mirror and telescope.

The sensitiveness of the apparatus was determined by finding the elastic constant of the suspension in a separate vibration experiment. The torque that ought to be exerted upon the condenser by the field was then calculated upon the assumption that the torque is equal to the angular rate of decrease of the magnetic energy.

The principal observations were taken during ten days in March, at which time the horizontal drift through the ether was shown to be near a maximum if one included the supposed velocity of the solar system through the ether; three of the observations were taken at noon, when the effects of the earth's orbital motion considered by itself would be a maximum, the others at 3 or 6 P.M. The plane of the plates lay N.E.–S.W.

Eleven observations are reported. The calculated deflections as given by the authors ranged from +.8 to −2.6 cm. at 1 M. distance for the orbital motion alone, and from 0 to −6.8 cm. for the combined motion. The observed deflections ranged from −.12 to −.35 cm., without obvious correlation with other circumstances.

3. *Critique.* A study of the report leaves one with the conviction that the experiment was ably performed and that the data given are reliable. But two points in the interpretation are, at least nowadays, open to serious criticism.

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[2] H. A. Lorentz, K. Akad. Amst. Proc. 6, p. 830, 1904.
[1] Trouton and Noble, Phil. Trans. A, 202, p. 165, 1903.

It is regrettable that good observations were not made and reported for several different periods distributed throughout the year. It is certainly possible that the earth might be almost stationary in the ether during March in consequence of the sun's proper motion through the ether, which we have no really good means of estimating. Or, again, the earth's speed might have been considerable during the period of the observations and yet, as an inspection of the geometrical relationships shows, its component of velocity in a suitable horizontal direction, which would alone be effective in producing a twist about the vertical, might have been always comparatively small. Probably our conviction as to the correctness of the result will not be greatly shaken by this circumstance, especially since the final results were obtained only "after many months of experience" with the apparatus; nevertheless, in contrast, the Michelson-Morley experiment certainly gains considerably in conclusiveness through the avoidance of just this sort of doubt.

A more serious matter is that the authors omitted the "Roentgen current," hardly known in their day, in calculating the magnetic field in the condenser. For they put $H=4\pi\sigma w$ where w=velocity parallel to the plates and $\sigma=$ surface density of charge on the conducting plates (in electromagnetic units). But the convection of the apparent charges on the mica dielectric also contributes to the magnetic field, and in the opposite sense, making the resultant field $4\pi\sigma w/K$. This illustrates the fact that in a moving electrostatic system the magnetic intensity due to convection is proportional at every point to the electric intensity.

In consequence the authors overestimate the torque to be expected in the ratio K^2. But later they make a slip by substituting for μK in the numerator v^2 in the denominator, where v="velocity of electric propagation" in the dielectric, and then taking $v=3.10^{10}$ in their numerical calculation. This wrongly divides the deflection by K and leaves it only K times too great. Thus, taking $K=6$, the deflection to be expected becomes, for the orbital motion, from $+.13$ to $-.43$ cm., or almost within the range of the errors of observation.

III. Theory of a Moving Condenser

The theory of the experiment is presented in detail in Laue's "Relativitätstheorie," but he does not consider the effect of a material dielectric and he supposes the moving condenser to undergo the relativity contraction. The introduction of this contraction does not appreciably alter the result but it seems preferable to omit it in the present discussion since the object of our inquiry is the result predicted by the fixed-ether theory. Accordingly we shall omit the contraction and give an outline of the argument.

4. According to the *Maxwell-Lorentz* theory the fundamental equation for the calculation of all ponderomotive forces of electromagnetic origin is

$$\mathbf{f}=\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{H} \tag{1}$$

where $\mathbf{v}$ denotes velocity of the electricity relative to the ether. The field vectors are connected in turn with the electricity by the equations (the units being ordinary Gaussian)

$$\left.\begin{aligned} curl\,\mathbf{H}&=\frac{1}{c}\left(4\pi\rho\mathbf{v}+\frac{\partial\mathbf{E}}{\partial t}\right). & div\,\mathbf{H}&=0. \\ curl\,\mathbf{E}&=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}. & div\,\mathbf{E}&=4\pi\rho. \end{aligned}\right\} \tag{2}$$

These equations lead, by a purely mathematical calculation, to the following well-known principle:

(A) The moment about any fixed axis of all the forces exerted by the field upon electricity equals the rate of decrease in the total moment about that axis of the electromagnetic momentum in the field, the electromagnetic momentum per unit volume being defined as

$$\mathbf{g}=\frac{1}{4\pi c}\mathbf{E}\times\mathbf{H}. \tag{3}$$

A special case (axis at infinity) is this:

(A') The vector sum of the forces exerted by the field upon electricity equals the rate of decrease of the total vector electromagnetic momentum in the field.

For *convection of an electrostatic system* we easily find further that

$$\mathbf{H}=\frac{1}{c}\mathbf{u}\times\mathbf{E} \tag{4}$$

where $\mathbf{u}$ = velocity of convection: while the problem of finding $\mathbf{E}$ can be reduced to an electrostatic problem by the following device:

(B) Imagine the moving system S, including its distribution of electricity, stretched uniformly in the direction of motion in the ratio γ and then brought to rest, forming what we shall call the "associated stationary" system S_1. Then the plot of the lines of $\mathbf{E}$ simply stretches into the plot of lines in S_1, and, letting $\mathbf{E}_1$ and V denote electrostatic intensity and potential, respectively, in S_1 and taking the x-axis parallel to the direction of motion, we have the following relationships between magnitudes at corresponding points in the two systems:

$$\varphi=\gamma V,\ E_x=E_{1x},\ E_{y,z}=\gamma E_{1y,z} \tag{5}$$

where φ = retarded scalar potential in S.

From this we find easily that

$$\mathbf{f} = -\nabla\psi,\ \psi = (1-\beta^2)\varphi = V/\gamma, \tag{6}$$

$$f_x = E_x = E_{1x},\ f_{y,z} = (1-\beta^2)E_{y,z} = 1/\gamma\ E_{1y,z} \tag{7}$$

where ψ is the "convection potential" and **f**, the force per unit charge.

Clearly the condition for electric equilibrium of a conductor is that in S ψ must be uniform, and in S_1, V; but (6) shows that both conditions are of necessity met simultaneously. Accordingly:

(*C*) To find the distribution of given charges which will be in equilibrium on a set of conductors in S, we have only to find the distribution of the same charges in electrostatic equilibrium in S_1 and then contract everything in the ratio $1/\gamma$ parallel to the motion.

Proofs of these principles and formulas may be found in Abraham's *Theorie der Elektrizität*, Vol. II, or in Richardson's *Electron Theory of Matter*.

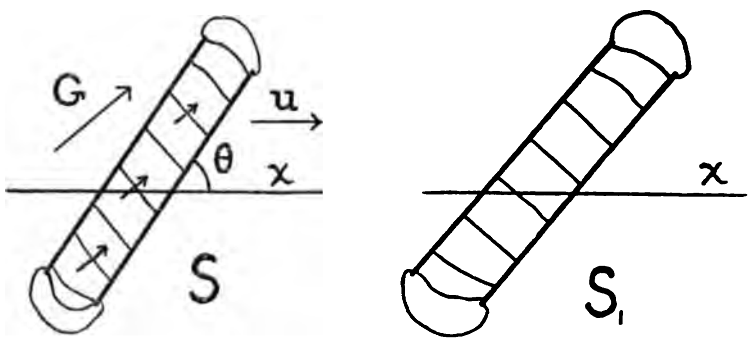

FIG. 1.

5. *Condenser with Ether as Dielectric*: β *small.*—Applying (B), we have as S_1 a stationary condenser that is slightly warped but with the plates still parallel. The warping slightly distorts the field in S_1; and then when we contract it to obtain the electric field in S, the lines become slightly inclined to the plates as shown. Accordingly, (3) and (4) show that **g** is inclined to the plates at a small angle (as shown by the arrows in the figure). But since the contraction is of order β^2 we shall incur relative errors only of the same order if we assume that the electric intensity is the same in the moving condenser as if there were no motion. **g** is then practically parallel to the plates and (3) and (4) give for its magnitude

$$g = \frac{\beta}{4\pi c} E^2 \cos\theta \tag{8}$$

where θ = angle between the plates and the direction of motion. The total momentum is therefore

$$G=\frac{\beta \cos \theta}{4\pi c}\int\int\int E^2 d\tau \text{ or } G=\frac{2\beta}{c}U \cos \theta, \tag{9}$$

where U = electric energy in the ether, the direction of **G** being practically parallel to the plates as shown in the figure.

A further slight error is introduced by the edge-field but this can be reduced indefinitely in the usual way by approximating the plates. The calculation given in Laue differs by a further small quantity of order β^2 because there the system S is supposed to undergo the Relativity contraction.

The *mechanical reactions* of the field upon the condenser are now readily obtained.

When the condenser is charged, (A′) shows that it will experience a backward impulse (the FitzGerald drag) nearly parallel to the plates and equal to $(2\beta/c)\ U \cos \theta$. For the earth's orbital motion $\beta=10^{-4}$, so that the maximum impulse would be less than $U\cdot 10^{-14}$. With modern apparatus a linear impulsive velocity of 10^{-3} cm. per sec. ought to be measurable to 10 per cent; but then $10^{-3}m=U\cdot 10^{-14}$, $U/m=10^{11}$, or we should have to load the condenser with 10^4 joules of electrical energy per gram of weight. At present this is probably impossible.

A more important conclusion for our present purpose is that the condenser also experiences a torque. For, taking a fixed axis at right angles to **G** and to the direction of motion (normal to the paper in the figure), we see that, since **G** is being carried along with a transverse component of velocity $u \sin \theta$, the moment of **G** about this axis is steadily increasing at the rate $Gu \sin \theta$ or, by (9), $2\beta^2 U \sin \theta \cos \theta$; hence by principle (A) there is a torque on the condenser equal to

$$L=2\beta^2 U \sin \theta \cos \theta; \tag{10}$$

and its direction will clearly be such (clockwise in fig.) as to tend to set the plates parallel to the motion.

This result confirms Trouton and Noble's calculation for the case $K=1$ except that the direction of the torque is reversed. The torque is small but ought to be readily detectable.

As a check, we shall attempt to calculate the torque directly from (1) and (4). At any point between the plates $H=\beta E \cos \theta$; as we pass through the charged layer on one plate the magnetic intensity decreases in proportion to the amount of charge passed over and becomes zero at a point outside of the charged layer. Hence one easily finds for the force per unit area on the plate $(H/2)(\sigma u/c)$ or, since $\sigma=E/4\pi$, $(\beta^2 E^2$

$\cos\theta)/8\pi$, the force being perpendicular to the direction of motion. Pairing off opposite elements of the two plates, we find a torque upon them equal per unit area to this force times the distance between the plates times $\sin\theta$, and summing for the whole condenser we arrive at just half of L as given by (10).

The discrepancy is not caused by the electric intensity, for this can be seen to exert no torque if we resolve the field into the elementary fields due to the separate elements of charge. In each of the latter fields the electric intensity points away from the element producing it and the law of action and reaction consequently holds for the interaction due to this cause between the members of any pair of elements.

As a matter of fact, the discrepancy is due to an excess force on the edges of the plate whose moment increases with the size of the plates and hence cannot be made negligible by increasing indefinitely the ratio of area to plate distance; details of this, slightly modified by the relativity contraction, are given in Laue's *Relativitätstheorie.*

6. *Effect of a Material Dielectric of Unit Permeability.*—This deserves careful consideration because mica condensers were used in the experiment.

Part of the theory can be extended to this case simply by including the polarization electricity in our distribution of charge. For then principles (A), (A′), (B), and equations (1)–(2), (4)–(7) will still hold and not only for the values at a point but also for the observable values of the vectors and the density, which represent mean values taken throughout a "physically small" volume. But (3), which is not linear, and (C) require elaboration.

The polarization **P** in S, being measurable as electricity displaced across unit area, stretches into a polarization $\mathbf{P}_1$ in S_1 given by

$$P_x = P_{1x}, \quad P_{y,z} = \gamma P_{1y,z} \tag{11}$$

Now if the dielectric were isotropic in both systems **P** would be parallel to **f** and $\mathbf{P}_1$ to $\mathbf{E}_1$; but then (11) would be inconsistent with (7). Hence we shall suppose an isotropic dielectric in S to become anisotropic in S_1, the polarization constant, $(K\text{-}1)/4\pi$, being decreased in a direction transverse to the motion in the ratio $1/\gamma^2$, so that

$$K_{1x} = K, \quad K_{1y,z} = K - \beta^2(K-1) \tag{12}$$

The polarization will then be in equilibrium in both systems simultaneously, and it follows that principle (C) can be extended to this case. The slight distortion of the field produced by the anisotropy in S_1 is easily seen to be of no appreciable consequence in the present connection.

Finally, in (3) let us split **E** and **H** into two parts and write $\mathbf{E} = \mathbf{E}' + \mathbf{E}''$, $\mathbf{H} = \mathbf{H}' + \mathbf{H}''$, where the plain letters denote values at an actual point and the single-primed letters, the observed mean values; then on the average $\mathbf{E}'\times\mathbf{H}''=\mathbf{E}''\times\mathbf{H}'=0$, and the mean momentum per unit volume is

$$\mathbf{g}'=\frac{1}{4\pi c}\left[\mathbf{E}'\times\mathbf{H}'+\mathbf{E}''\times\mathbf{H}''\right]. \tag{13}$$

The first term on the right leads at once to equations (8)–(10), that is, this term gives, for a fixed potential difference between the plates, the same value for the torque as if the material dielectric were removed. This is the result to which we are led if we follow the usual rule of simply ignoring fine-grained irregularities of field.

7. *Local Torque in a Dielectric.* The second term on the right in (13) requires special consideration. The general result appears to be that it yields an additional torque which may greatly exceed the one given by (10) but which can have either sign and which has nothing to do with the ordinary electrical properties of the substance but depends directly upon its atomic structure. Indeed, one would expect a torque to exist in many crystals even in the absence of an electric field.

We shall illustrate the possibilities of the case by considering a simple group of electric doublets whose dimensions are small relative to their distance apart.

The mechanical forces upon a given doublet arise in part from the field of its neighbors and of distant charges. This part will depend only upon the strengths and positions of the doublets and of other charges.

A second part arises from the magnetic interaction between the constituent charges of each doublet. If the latter are $+e$ and $-e$ and are separated by a displacement l making an angle ζ with the x-axis, which we suppose to be the direction of motion, the doublet experiences a torque tending to set its axis at right-angles to the direction of motion and of magnitude (the magnetic intensity being $\beta e \sin\zeta/l^2$).

$$L_1=\frac{\beta^2e^2}{l}\sin\zeta\cos\zeta. \tag{14}$$

Let there be N doublets per unit volume with parallel axes. Then the torque per unit volume is $L=NL_1$, while the resulting polarization is $P=Nel$, so that the torque per unit volume can also be written

$$L=\frac{\beta^2P^2}{Nl^3}\sin\zeta\cos\zeta. \tag{15}$$

This torque can attain any magnitude independently of P through variation in l. Hence a counterbalancing of it by other effects which depend upon P can occur only as a matter of accident. The denominator Nl^3 is, according to our assumptions, much less than unity.

From this result one seems justified in drawing the broad conclusion that *crystals* ought, in general, to experience an appreciable torque when moving through the ether. Definite calculations cannot be made without the adoption of some definite atomic theory, but all modern theories suppose electrical separations to exist in the atom which by themselves would produce enormous values of P, and in general one would expect these to give rise to an outstanding torque of the magnitude of L as given by (15) with P a large number and Nl^3 at least less than unity. For instance, if 10^{23} electrons per unit volume were displaced relative to the atoms a distance of only 10^{-10} cm. the resulting polarization would be $10^{23} \cdot 10^{-10} \times 4.77 \cdot 10^{-10} = 4{,}770$ electrostatic units. This value of P gives, even with Nl^3 replaced by unity, a value of L which is hundreds of times bigger than the torque per unit volume on Trouton and Noble's condenser; for in the latter case the electric intensity appears not to have exceeded 700 electrostatic units, so that U in (10) divided by the volume would be about 20,000.

Since the occurrence of such a torque would constitute an effect of uniform motion through the ether, Relativity requires that any torque due to this cause must be compensated within the crystal by an equal and opposite torque of different origin (presumably of the same nature as that described below in Sec. 10). Thus the fact that no such torque has ever been observed lends a certain amount of support to Relativity.

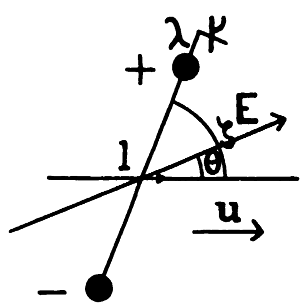

Fig. 2.

To illustrate now the possibilities in an *isotropic* medium, let us consider the effect of applying an electric field in a direction inclined at an angle θ to the direction of motion (Fig. 2), and let us suppose the doublets to be oriented irregularly except that their axes are all parallel to the plane defined by these two directions. Let the field displace the positive charge a small distance λ in the direction of the axis and a small distance μ at right angles to it where

$$\lambda = aE\cos(\zeta - \theta), \quad \mu = bE\sin(\zeta - \theta). \qquad (16)$$

In this case there is an outstanding second-order torque for which, using (14), I find the value

$$L=\frac{N\beta^2e^2}{16l^3}(2a^2-8ab+5b^2)E^2 \sin 2\theta,$$

or, introducing $P=Ne(a+b)E/2=(K-1)E/4\pi$,

$$L=\left[\frac{(K-1)^2}{8\pi Nl^3}\frac{2a^2-8ab+5b^2}{(a+b)^2}\right]\frac{\beta^2E^2}{8\pi} \sin 2\theta, \tag{17}$$

where L = torque per unit volume.

This expression is the same as (10) divided throughout by the volume of the condenser, except for the factor in brackets. It is hard to assign a plausible value to the latter factor without assuming some definite theory of atomic structure; but according to our assumptions Nl^3 is small compared with unity, hence the whole bracket ought for a solid dielectric to be comparable with unity and might greatly exceed this value, and it might have either sign. For instance, for $Nl^3=1$, $K=6$ (mica) and $a=0$ (approximately pure rotation of the doublets) the bracket equals $+5$, while for $Nl^3=1/10$, $K=6$ and $a=1.5\ b$ the bracket equals -4.

In an actual dielectric, circumstances will no doubt be very different from those of the simple case here treated. Yet we seem justified in concluding that, according to the electromagnetics of Lorentz, a local torque of appreciable magnitude is very likely to act upon a moving polarized dielectric and that this torque might conceivably mask completely the effects of the torques upon the true and apparent charges.

8. *The Effect of the Mica in Trouton and Noble's Condenser* remains therefore in doubt. It may have caused the null result; more likely, however, being crystalline, it should have greatly increased the effect, perhaps with a reversal of sign.

9. *The Fine-Structure of the Charge on the Plates* might conceivably have a similar effect. The removal or addition of electrons on the surface might form doublets with axes more or less parallel to the surface, and these, by urging the plates toward a position at right angles to the motion, might mask the main effect. Such an effect could be distinguished in a repetition of the experiment through the circumstance that it would depend only on the charge and not, like the main torque, also upon the difference of potential.

10. *The Explanation by Relativity* of the null result is a dynamical one and is fully given in Laue's *Relativitätstheorie*. The torque occurs as a secondary effect superposed upon the far larger electrostatic attraction between the plates: according to Relativity, the intermolecular stresses which balance the latter attraction do not obey Newton's

Third Law but themselves produce a torque which just balances the electromagnetic one. The null result is thus explained, not exactly by the Lorentz-FitzGerald contraction itself, but rather as a consequence of the same cause that produces this contraction.

11. *Other Ways of Excape* are hard to find. So far as the writer is aware, there is no rival to the Maxwell-Lorentz theory which explains all ordinary phenomena (including Hertzian waves) and also removes the torque on the condenser. Of course, drag of the ether by the earth would do it, but this assumption leads to well-known difficulties. Instead of speculating upon possible modifications of electromagnetic theory it seems more profitable to pass in review the experimental facts upon which the prediction of the torque rests. They are (for vacuum as dielectric):

(1). Moving charged bodies generate a magnetic field. The charged body used in experiments like Rowland's is very similar to either plate of our condenser, the only important difference being that in those experiments the average convection current was closed.

(2). Magnetic fields (of certain kinds, at least) act on moving charged bodies (probably verified only for charged molecules or electrons).

(3). No difference has yet been detected between magnetic fields arising from different causes.

These basic facts lead by very simple reasoning to the predicted torque. As a matter of logic one might attack the sufficiency of the present experimental basis for (2) and (3), but the prospect of discovering any error at this point seems very small. Perhaps the most promising thing to try out would be whether a moving charged *conductor* really is acted upon by a magnetic field.

12. *In Conclusion*, the situation may be summed up as follows:

Trouton and Noble's negative result might have been due to either (1) insufficient sensitiveness of their apparatus (Sec. 5), or (2) insufficient distribution of their observations over different times of year (Sec. 5), or (3) a special effect of the mica dielectric (Sec. 7, 8), or (4) a similar special effect in the charged surface of the plates (Sec. 7, 9).

If, however, it be accepted as an experimental fact that an air condenser never experiences a torque due to the earth's motion, then this fact speaks forcibly in favor of Relativity and can hardly be explained on any other basis.

Probably few physicists will refuse today to accept this as an experimental fact, nevertheless the importance of the problem seems to justify a repetition of the experiment, with observations taken on an air condenser, at different times both of day and of year, and with various distances between the plates.

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CELESTIAL MECHANICS

A Survey of the Status of the Determination of the General Perturbations of the Minor Planets

BY

A. O. LEUSCHNER

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CELESTIAL MECHANICS

A SURVEY OF THE STATUS OF THE DETERMINATION OF THE GENERAL PERTURBATIONS OF THE MINOR PLANETS.

*Appendix to the Report of the Committee on Celestial Mechanics, National Research Council**

BY A. O. LEUSCHNER

Professor of Astronomy, University of California

CONTENTS

*This Committee of the Division of Physical Sciences of the National Research Council consists of the following members: E. W. Brown, Professor of Mathematics, Yale University, Chairman; G. D. Birkhoff, Professor of Mathematics, Harvard University; A. O. Leuschner, Professor of Astronomy, University of California; H. N. Russell, Professor of Astronomy, Princeton University.

INTRODUCTION.

With approximately one thousand asteroids discovered and believed to be sufficiently observed to permit of fairly reliable orbit determinations, as indicated by the permanent numbers assigned to them, the task of preserving these discoveries has grown so stupendous that the time seems to have arrived for an analysis of the present astronomical practice in providing the necessary additional observations and calculations.

Hitherto, the burden of correcting orbit elements and computing ephemerides has rested principally on the Berlin Recheninstitut. In recent years the Marseilles Observatory has rendered notable service in contributing orbits and ephemerides. Observations, photographic and visual, are regularly made at a number of observatories. The Berlin Recheninstitut publishes ephemerides and other results in the Astronomische Nachrichten, and in the Ephemeriden der Kleinen Planeten. Up to 1918 these data appeared also in the Astronomisches Jahrbuch. The number of oppositions during which the minor planets have been observed, and the status of orbit determinations are annually summarized in the Vierteljahrschrift der Astronomischen Gesellschaft. In 1901 Bauschinger published the latest reliable elements, etc., with data concerning the perturbations for the then known 463 planets, "Tabellen zur Geschichte und Statistik der Kleinen Planeten."

The latest available collection of elements is contained in the Berlin Jahrbuch for 1918. The adopted Jahrbuch elements serve the purpose of providing ephemerides from opposition to opposition. Their origin may be traced from the notes given from year to year in the Jahrbuch, ending with 1918, and in *Kleine Planeten.* Some of the elements include arbitrary corrections to the mean motion and to the mean anomaly for the purpose of representing late oppositions so as to serve for prediction of immediately following oppositions. In other cases, approximate or accurate perturbations are included, with or without correction of the elements by the usual least squares adjustment. For thirty-six planets the elements in the Jahrbuch of 1918 are mean or osculating elements, derived in connection with general perturbations which are approximately included in the prediction of ephemerides. Until similar fundamental data shall have become available for the remaining planets, the present practice of the Recheninstitut appears to furnish the only certain method for the preservation of planetary discoveries.

In addition to the general pertubations of the thirty-six planets which are being used by the Jahrbuch, the general pertubations of a number of other planets have been derived on the basis of what, at the time, appeared to be reliable osculating elements. These presumably valuable data have been replaced by later elements derived independently, more or less accurately, with or without general perturbations, or upon the basis of arbitrary corrections. Bauschinger's Tabellen form a valuable key to some of these investigations, but even in the Tabellen the elements and perturbations cited do not, in all cases, represent the best elements and perturbations available, although perhaps in every case they are the most reliable for subsequent oppositions. This arises from the fact that earlier investigations were abandoned by Bauschinger in favor of later ones. Preliminary calculations have shown, however, that some of the earlier elements and perturbations represent distant oppositions at a later date more satisfactorily than his adopted elements and perturbations represent earlier oppositions equally remote.

Of great importance for the program of the Recheninstitut are the contributions of Brendel, who has developed methods for the approximate determination of the perturbations for certain groups of planets. Perturbations greater than 3′.4 within fifty years are included, with the object of reproducing geocentric places within 20′ for 100 years. So far the necessary data have been published by Brendel, Labitzke, and Boda for 230 planets, approximately 25 per cent of the total number of known minor planets. The advantage to be gained from Brendel's contributions for these planets is that for the practical purpose of preserving these planets, by following their motion, it should become unnecessary as a rule to compute special perturbations for them, or even to apply corrections to the elements. Brendel plans to continue the work of supplying instantaneous elements and approximate perturbations for other groups of planets so that the program of the Recheninstitut, of the Marseilles Observatory, and of various investigators who, from time to time, publish improved elements and perturbations for ephemeris purposes, will become more and more simplified.

The preservation of planetary discoveries by observation and prediction with the aid of approximate perturbations is not the ultimate aim of astronomical science, but a necessary and unavoidable means to the end. The ultimate aim rests on the determination of mean elements and general perturbations which hold for all time or at least for very long periods within the limits of accuracy set by observation. It is expected that the elements and perturbations determined under the Newtonian law of gravitation may serve this purpose, provided that the mathematical difficulties will not prove insurmountable. It may be assumed that the rigid mathematical methods hitherto developed are satisfactory for planets with moderate eccentricity and inclination which are not in a very near commensurable ratio with any of the major planets, but it has not been established so far whether an accurate application of the Newtonian law would fully account for the motion of the minor planets even in the ordinary cases just referred to.

Exhaustive researches are available only for a very limited number of planets. Among these are (4) Vesta, (13) Egeria and (447) Valentine. The researches on (4) Vesta are due to Leveau, whose extraordinary investigations extend approximately over a complete century of oppositions. In connection with his work on the motion of Vesta, Leveau has aimed at a determination of the masses of Jupiter and Mars. His final value is larger than the best available mass of Jupiter by approximately one one-thousandth. On account of the moderate perturbations, the motion of Vesta does not lend itself as well to a determination of the mass of Jupiter as the motion of other minor planets with very large perturbations. Any slight departure from the true mass of Jupiter, et cetera, can reveal itself through the motion of Vesta only in long intervals of time, which accounts for Leveau's gradual improvement of his adopted mass by successively including longer periods of observation. For the present his results may be considered fundamental and final, so far as this planet is concerned. No other case has been studied so exhaustively. Later predictions are well within the errors of observation, and not the slightest departure from the Newtonian law is noticeable. It remains, however, to establish the same result for planets with large perturbations, particularly for such planets as have a mean motion commensurable with that of Jupiter. To avoid the necessity of gradually improving the Jupiter mass by means of subsequent observations of Vesta, it appears advisable to base further predictions on the best determined values of the masses of the major planets. Vesta also furnishes an example of the weight to be assigned to observations in the early part of the last century.

Leveau's investigations furnish a striking example of the fundamental researches necessary for the promotion of astronomical science as distinguished from the generally accepted program of observation and prediction for the preservation of discoveries.

For the study and interpretation of planetary statistics, particularly with reference to the origin of minor planets, the explanation of the gaps, the question of stability and ultimate destiny, or in general regarding their place in any hypothesis concerning the solar system, final mean elements derived on the basis of accurate developments of the perturbations are most essential. Fragments of fundamental investigations of perturbations are available for a number of minor planets. The value of some of these has been vitiated by corrections made in connection with the accepted program of approximate prediction, such as, for example, the correction of an accurate set of osculating elements derived by special or general perturbations, to represent later oppositions, either without perturbations or by taking account only of approximate or incomplete perturbations.

Fundamental investigations here are understood to include the determination of osculating or mean elements from a limited number of oppositions with complete regard of the perturbations, either special or general, in so far as they may have been appreciable. In connection with the study of the data existing for a limited number of selected planets, it has been found that the failure of such elements and perturbations to represent future oppositions in some cases can be accounted for by the fact that the masses of the major planets were known at the time with insufficient accuracy. The mere correction of the perturbations therefore, for the latest known values of the masses may render such elements and perturbations far more satisfactory than they appeared to be at the time when they were discarded in favor of new determinations of elements with or without perturbations.

Freed from effects of changes which affect disadvantageously their permanent value the fragments of fundamental investigations referred to are of great importance as a basis for researches and their intelligent application will involve a vast saving in computational and theoretical work.

At present it appears next to hopeless to the investigator to adopt a profitable form of attack in connection with any of the older minor planets without an enormous expenditure of time in searching astronomical records. This accounts for the many duplications of effort and for the disregard of previous valuable investigations. If systematically undertaken, the task of bringing to light the important data available for a final determination of the elements and general perturbations of the minor planets, does not appear insurmountable. Once available, such research surveys will be invaluable and should prove an encouragement to research, particularly to young investigators.

The research surveys of the few planets which are given below are intended to serve as illustrations of the data which should be made easily accessible. No claim is made for the absolute completeness of these data. The time for active work, with the aid of a few assistants, to prepare these preliminary surveys has extended only over a little more than a month. A great mass of material had to be consulted which was found to be of no importance to the purpose in hand. This is being preserved on cards for easy reference, if required at any time. Thus care has been taken to eliminate elements which would not be considered as fairly accurate osculating elements particularly those which have resulted from corrections on the basis of subsequent oppositions purely for ephemeris purposes, without complete consideration of the perturbations and of the earlier oppositions in the final adjustment. This policy, however, has not been adhered to strictly, partly for historical and theoretical reasons with reference to preliminary elements, and partly for other reasons with reference to later elements.

Whenever possible, the reasons for the abandonment of previous investigations are given, but in many cases no reasons could be found, at least not in the astronomical records available in the library of the University of California. Some of these reasons are probably to be found in the records available in the library of the Lick Observatory, but in the limited time it has not been possible to consult these or other additional records for this preliminary survey. An immense amount of fundamental work has been accomplished by the Berlin Recheninstitut, particularly in computing special perturbations and deriving osculating elements, but has been published only in part. The remainder reposes in the archives of the Recheninstitut. It may be assumed that the immense task of providing ephemerides has interfered with the publication of the accumulated material. Without this material, research surveys such as those presented here are not complete.

A simple way of accomplishing the introduction of the improved mass of disturbing planets referred to above, is to multiply the final sum of all the terms for each component of the perturbations by the ratio of the new to the old mass. Aside from the improvement which it may be possible to make to some of the older fundamental data, particularly those which are no longer used for ephemeris purposes, by the introduction of the best determined values of the masses of the major planets, it is probably possible to enhance their values still further by correcting the elements on the basis of the existing developments of the general perturbations, with the aid of subsequent oppositions and in case of appreciable changes in the elements, by also correcting the numerical coefficients in the general developments by differential methods.

After revision of independently determined elements and perturbations for separate series of fairly consecutive oppositions, disconnected by a gap including a number of oppositions to which neither series was extended either backward or forward, the separate fundamental investigations will, in some cases, probably be found to be entirely consistent and thus become of permanent value, such as Leveau's investigations on (4) Vesta, without involving extensive theoretical and numerical work. In other similar cases the correction of the elements and perturbations pertaining to fundamental investigations for groups of oppositions separated by considerable gaps, so as to represent the osculating data at a subsequent epoch, may establish satisfactorily the connection between one or more groups of oppositions for which elements and perturbations have been independently determined with accuracy. The mode of attack will vary with the available data for different planets, as indicated by the research surveys, which this discussion advocates. The resurrection of the classical contributions of the pioneer investigators of planetary perturbations on a permanent basis, should produce material of great value for the ultimate aims of astronomical science concerning planetary investigations.

The proposed program of fundamental investigations cannot supersede the present astronomical practice in caring for the minor planets in the immediate future, but as stated above it will be of great assistance for the practical purposes of prediction, and should gradually solve the now stupendous task of preserving planetary discoveries, while furnishing at the same time the data for the more fundamental aims of astronomical science.

For the majority of the minor planets, probably the application of four successive steps or processes will be necessary to preserve the discoveries until final elements and perturbations can be made available. The first step or process represents the present practice principally conducted by the Berlin Recheninstitut. The second step is illustrated by Brendel's plan of supplying instantaneous elements and approximate perturbations. The third step corresponds to the determination of the elements and perturbations of the Watson asteroids undertaken by Leuschner, which are intended to provide fairly accurate but not final results. Hansen's and the Bohlin-v. Zeipel methods have been found most practical and accurate in this connection. The fourth and final step is demonstrated by the fundamental work of Leveau on (4) Vesta. It is the object of this discussion to encourage researches similar to Leveau's, and by supplying samples of research surveys for a limited number of planets to pave the way for a comprehensive international program in this connection.

It cannot be too strongly emphasized that accurate osculating elements are absolutely essential for fundamental investigations of the perturbations. While this requirement is fully recognized, the prevailing practice of changing elements for immediate ephemeris purposes is apt to lead to erroneous interpretation of available elements. Mean elements, in general, can be determined only after osculating elements and perturbations shall have become available. Some investigators have adopted as approximate mean elements the average of elements published for more or less extensive series of oppositions, assuming that these elements represent fairly reliable osculating elements. Even if this were the case, it hardly ever occurs that a sufficiently large number of elements, uniformly distributed over the orbit, are available to guarantee that, in taking the average, the effect of the periodic terms is entirely eliminated. But, as previously stated, many of the apparently reliable sets of elements are not osculating, but inferior elements produced by arbitrary changes or with incomplete perturbations.

Practically the only reliable method of arriving at accurate initial osculating elements consists in representing the observations of a limited number of oppositions by taking into account the special perturbations and in testing the validity of the resulting elements for one or more oppositions following. Osculating elements thus obtained will rarely require later changes which would affect the coefficients of the general perturbations. No correction of such elements should be attempted, except on the basis of the determination of complete special or general perturbations. As it was not considered necessary, at the time, to adhere strictly to the foregoing principle in Leuschner's program for the determination of the perturbations of Watson's asteroids, allowances for slight inaccuracies may later become necessary for some of the Watson planets. In particular, corrections to the larger coefficients of the perturbations may be necessary for the planets for which the initial adopted elements, considered at the time as sufficiently accurate, were neither accurate mean elements nor accurate osculating elements.

Attention has recently been called, in the Proceedings of the National Academy of Sciences of 1921, Vol. 8, No. 7, p. 170, and in the report of the Committee on Celestial Mechanics of the National Research Council, Bulletin of the National Research Council, Vol. 3, Part 4, No. 19, June, 1922, to the extremely satisfactory results obtained for the planets (10) Hygiea, and (175) Andromache, by the application of Leuschner's revision of von Zeipel's tables for the Hecuba group. Further reference to the great importance of "Gruppenweise Berechnung der Stoerungen," inaugurated by Bohlin, may therefore be omitted here. The methods of Bohlin and his followers serve admirably in connection with the third of the four stages outlined above for the determination of fundamental results. In certain cases of limited eccentricity and inclination, they will, no doubt, lead to final results.

No claim is made that the planets for which research surveys are given below are the ones most in need of immediate attention. Further study of available data will be necessary to classify the planets with reference to the requirements of observation and computation, as outlined in the report of the American Committee on Comets and Asteroids, presented at the Brussels meeting of the International Astronomical Union in 1919; nor are the planets considered below the most important for fundamental scientific purposes. The list, however, may be considered as fairly representative of the immediate research requirements. To some extent the selection has been accidental. Thus the computing section of the British Astronomical Society has undertaken the computation of the ephemerides of the first four planets. In this connection it sought advice regarding the best available data and methods of procedure. The research surveys of the first four planets were undertaken to aid the computing section in its undertaking. The importance of the Trojan group is too well known to be emphasized. For further investigations concerning the theories of the six planets belonging to this group the research surveys given will be of considerable value. It is of interest to note that Leuschner's orbit methods as applied by Einarsson, appear to be the most promising for the determination of preliminary osculating elements, while Wilkens' method deserves careful trial in deriving the perturbations. E. W. Brown's unpublished theory promises to be thoroughly fundamental. For the two planets of the Trojan group last discovered, more accurate preliminary osculating elements are immediately needed. For other planets, the list of research surveys themselves will reveal the most necessary work to be done. In general, reference to theoretical investigations is included only in connection with a simultaneous new determination of elements. Thus the numerous and important investigations on the theory of the Trojan group are not considered here, the chief object of the surveys of these planets being to furnish numerical data and encourage their improvement as a basis for such theories.

The form in which the research surveys are presented must be considered experimental. That adopted is the outcome of several other attempts at presenting the material. It is hoped that this report will call forth helpful criticisms and suggestions which may ultimately lead to the adoption of some definite plan for international cooperation. Much material has been collected on planets not included in the list, which, it is hoped, may be printed later.

It was found that the research surveys for the various planets could not be made so complete that the investigator may abstain from referring to the sources themselves. This applies also to the collection of elements. The elements are collected merely for purposes of comparison and are not reproduced with uniform accuracy.

For practically all the planets in the list, except the first four and several others, a fairly complete bibliography of observations has been prepared, but this bibliography is published here only for the last two of the Trojan group.

Attention might well be called here to the need of curtailing indiscriminate observations. Even in recent years observations have been multiplied for planets for which two or three accurate observations at each opposition would be sufficient for all scientific purposes. It is planned to formulate in the near future definite proposals for an international program of observations.

The main purpose of this report is the encouragement of fundamental researches essential to the ultimate aims of astronomical science, which, for their consummation, require the knowledge of accurate elements and perturbations of the minor planets.

The surveys have been prepared in the main by Dr. W. F. Meyer, and by Dr. H. Thiele, assisted by several advanced students in astronomy, who have gathered the necessary references. For the Trojan group, unpublished data collected by Dr. Sturla Einarsson have been available.

As a rule the abbreviations adopted for the references are those of the Astronomischer Jahresbericht.

The usual notations of the elements are adhered to, both μ and n being used for the mean daily motion.

(1) CERES.

The first and largest of the minor planets was discovered 1801, January 1, by Piazzi in Palermo.[1]

Piazzi assumed that the object was a comet, but several astronomers succeeded in proving from the 22 meridian observations near the stationary point over an heliocentric arc of 9° that it was a planet moving in a nearly circular orbit; thus Burckhardt[2] computed Elements A, Olbers[3] the circular Elements B, Piazzi[4] the circular Elements C. Only the computation by Gauss,[5] Elements D, was accurate enough, especially in the determination of perihelion and eccentricity, to indicate where the planet might be found the following year.

Olbers found Ceres again 1802, January 1, ½° from the predicted place, near the place where, three months later, he discovered the second of the minor planets. The new observations naturally increased the accuracy of the elements notably; thus Gauss[6] computed Elements E, from observations in 1801, and January 1802; representation in February 1802, +7″ in α, —20″ in δ. Burckhardt[7] including the perturbations larger than 30′ found Elements F.

For some years the orbit of Ceres was investigated by Oriani, Burckhardt, and Gauss by taking the perturbations into account, but the efforts of Gauss went farther than those of the others. Burckhardt[8] started with the computation of perturbations at intervals of two days, and later computed tables founded upon them. Oriani[9] used Laplace's method, with which also Gauss started. Gauss developed the perturbations first in 1802, together with Elements VIII, G, and formed tables of perturbations[10] and later in 1805[11] when he used the same interpolatory development of the perturbative function as Hansen later used in 1830.

The orbit computation was taken up later by Heiligenstein.[12] He derived Elements H from the oppositions 1818, 1820, 1821, 1822, 1825, 1826, 1827, with special perturbations of the elements by Jupiter, (mass 1/1053.924). Representation of the normal places —10″ to +6″ in mean longitude. Correction to the ephemeris for 1830 April, May, —6″ in α, —10″ in δ.

Heiligenstein's ephemeris deviates 15′ from the ephemeris in B. J. 1830, which was based on the elements of Gauss (XIII, 1809), using the tables of perturbations by Gauss and an empirical correction by Encke of 14′ to the mean longitude determined from the last observations.[13]

In B. J. for 1831 Encke[14] gives an ephemeris from new Elements I of his own based on the oppositions 1820, 1821, 1822, 1825. Jupiter mass 1/1053.924. Special perturbations by Jupiter only. Representation:

	1820	1821	1822	1825	1827	1829
in α	—6″	+2″	—4″	—3″	—2″	—27″
in δ	0″	0″	+6″	+1″	0″	11″

In B. J. 1832 to 1836 the ephemerides by Heiligenstein were published. Later the computation by Encke and Wolfers was used to 1871.

In the meantime Damoiseau[15] had given expressions for the perturbations containing a large number of terms but the individual coefficients do not seem to be very exact, according to Hill.

For the use of the American Ephemeris, E. Schubert[16] undertook to correct the elements by 250 observations in 14 oppositions, 1832-1854, using special perturbations of the elements by Jupiter as computed by Encke and Wolfers but corrected for the secular variation of the obliquity.[17] Elements J. Residuals in α —22″ to +21″, in δ —8″ to +8″; ϕ corrected according to A. J., Vol. 5, p. 73. A further correction of the elements by Schubert[18] was based on only four normal places in 1853, 1854, 1855, 1857; he applied the special perturbations of the elements by Jupiter and Saturn. Representation of the normals ±0″, "by which the correctness of the whole is proved." Elements K.

Godward[19] repeats the process of Heiligenstein, Encke, Wolfers, Schubert. The errors for fifteen oppositions 1857 to 1876 of the ephemerides in Nautical Almanac which include the perturbations of Venus, the Earth, Mars, Jupiter, Saturn gave by a least squares solution the Elements L. Ephemerides by these elements were given in the Nautical Almanac to 1913.

The corrections to Encke's ephemerides increased after 28 years to $\pm 3^s$ in α ±20″ in δ.

The corrections to Schubert's ephemerides increased after 23 years to $+6^s$ in α ±40″ in δ.

The corrections to Godward's ephemerides increased after 36 years to $+2^s$ in α ±10″ in δ.

For the purpose of illustrating his modified form of computing absolute perturbations Hill[20] computed the first order perturbations of Ceres by Jupiter starting with the first elements by Schubert (uncorrected). It was found that the osculating mean motion differed widely from the mean mean motion. An arbitrary value was substituted. The Jupiter mass is taken to be 1/1047.355. The expressions for the periodic terms of the perturbations are given. In order to arrive at mean elements as well as to see how closely the perturbations represent the observations, ten normal places—1802, 1807, 1830, 1857, 1863, 1866, 1873, 1883, 1885, 1890, were formed. Secular perturbations of Mars, Jupiter, and Saturn were computed by the method of Gauss. The periodic perturbations by Mars and Saturn were taken from the tables of Damoiseau. Preliminary elements and a least squares solution led to mean Elements M. The residuals are —40″ to +40″ in hel. longitude, —20″ to +13″ in geoc. latitude. Hill originally intended to enlarge and complete his theory of Ceres; for this purpose he collected the observations into 75 normals from 1801-1897.[21] He published the positions because he did not expect to finish the work. The collection is not complete.

As an extension of Hill's work Merfield[22] has given a computation of the secular perturbations of Ceres arising from the action of the eight Major Planets. From Hill's theory and his mean elements using the method of Gauss as set forth by Hill the numerical values of the action of the planets were derived.

M. Wolf[23] has developed the expression (ρ) according to the theory of Gyldén in the case of Ceres. Cf. Tisserand, Mécanique Céleste, Vol. 4.

M. Viljev[24] has published tables of absolute perturbations of Ceres after the method of Hansen.

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[13] B. J. 1830, p. 245.

[14] B. J. 1831, p. 277. A. N. vol. 27, p. 177.

[15] Connaissance des Temps 1846. Additions p. 32. Not available here.

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[20] A. J. vol. 16, p. 57.

[21] A. J. vol. 21, p. 51.

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TABLE 1.—*Elements—(1) Ceres*

Letter	Date	MT	L	π	☊	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A....	1801 Jan. 1.3328	....	68 59 37	248 59 37	80 58 30	10 47
B....	1801 Jan. 1.....	....	68 35 51.5	....	80 22 45	11 3 36
C....	1801..........	....	68 46 41	....	80 46 48	10 51 12
D....	1801..........	Palermo....	76 28 14	150 33 20	81 2 35	10 36 30
E....	1801..........	Palermo....	77 27 31	145 57 15	80 58 40	10 37 57
F.....	1802..........	....	155 32 35	146 44 37	81 5 35	10 36 52
G....	1801..........	Seeberg....	77 19 34.9	146 33 37	80 54 59	10 37 56.0
H....	1818 Oct. 15.0..	Göttingen..	28 21 52.291	148 2 14.084	80 48 32.192	10 38 21.682
I.....	1822 Jan. 22.0..	Göttingen..	127 36 44.2	147 36 57.6	80 41 55.0	10 38 7.7
			M			
			° ′ ″			
J.....	1854 Jan. 0.....	Washington.	113 22 25.08	148 55 23.41	80 50 50.79	10 37 8.54
K....	1854 Jan. 0.....	Washington.	113 18 22.40	148 56 3.72	80 50 31.11	10 37 4.76
L.....	1854 Jan. 0.....	Washington.	113 22 11.73	148 55 26.54	80 50 31.06	10 37 5.81
			L			
			° ′ ″			
M....	1850 Jan. 0.0...	Greenwich..	309 30 32.4	148 28 32.5	80 48 5.6	10 37 6.2

Letter	Date	MT	φ	μ	Equinox	Author
			° ′ ″	″		
A.....	1801 Jan. 1.3328	....	2 5	859.05	....	Burckhardt I
B.....	1801 Jan. 1.....	....	0	786.528	....	Olbers
C.....	1801..........	....	0	795.937	....	Piazzi
D....	1801..........	Palermo....	4 2 45	784.254	....	Gauss I
E.....	1801..........	Palermo....	4 40 10	769.7925	....	Gauss VII
F.....	1802..........	....	4 31 25	771.363	....	Burckhardt II
G.....	1801..........	Seeberg....	4 31 17.8	770.7951		Gauss VIII
H....	1818 Oct. 15.0..	Göttingen..	4 31 5.183	771.2273825	1818.00	Heiligenstein
I.....	1822 Jan. 22.0..	Göttingen..	4 31 18.0	770.72468	1810.00	Encke
J.....	1854 Jan. 0.....	Washington.	4 24 28.41	769.63875		Schubert
K....	1854 Jan. 0.....	Washington.	4 24 29.36	769.62476	1854.00	Schubert
L.....	1854 Jan. 0.....	Washington.	4 24 29.65	769.64746	1854.00	Godward
M....	1850 Jan. 0.0...	Greenwich..	4 29 57.8	770.718276	1850.00	Hill*

*Mean elements.

(2) PALLAS

Discovered by Olbers at Bremen 1802, March 28.[1] Olbers attempted to compute a circular and a parabolic orbit for the new planet, both of which failed. His computation showed the orbit had a large inclination and considerable eccentricity.

From observations extending from April 1 to July 8, Gauss[2] computed Elements A (Gauss V). They are improvements on preliminary sets. With these elements an ephemeris for 1803 was computed.

From observations extending from April 4 to May 20, Burkhardt[3] computed Elements B. With these elements Burkhardt computed the perturbations in longitude, latitude, and radius vector covering the period April 4 to May 20.

The planet was reobserved by Harding 1803, Feb. 21st. The comparison between Gauss' ephemeris and observations was as follows:

1803	$\Delta\alpha$	$\Delta\delta$
Feb. 21	+2′ 02″	—34″
Feb. 23	+2 35	—57

On the basis of these residuals, Gauss[4] improved Elements A Gauss (V). These new Elements C (Gauss VI) represent the observations as follows:

1803	$\Delta\alpha$	$\Delta\delta$
Feb. 21	− 20″.0	+ 15″.8
Feb. 23	+ 7.8	− 7.7

From a set of elements, based on oppositions 1804, 1805, 1807, 1808, Gauss[5] derived an improved set of Elements D from a least squares solution. This solution includes also the oppositions 1803 and 1809 and forms the basis for the computation of perturbations as outlined below.

Gauss[6] first attempted to construct tables of perturbations for the four known minor planets, but the large eccentricity and inclination forced him to formulate a theory for Pallas based on the variation of the elements expressed analytically and integrated by mechanical integration. From two successive calculations of the special perturbations, due to Jupiter, Gauss derived the improved Elements E, which represented the heliocentric longitudes for the first seven oppositions within ±8″.

In 1811 Gauss[6] began his first computation of general perturbations due to Jupiter. For this purpose he used Laplace's elements of Jupiter, epoch 1805, and his own elements of Pallas for the same epoch. This computation led to a set of mean Elements (F). With these mean elements for epoch 1810 and similar elements for Jupiter (Laplace) a second computation of general perturbations due to Jupiter was undertaken. This computation led to the following results:

The mean motion of Pallas oscillates between 18/7 of ♃ motion $\pm 0''.2153$, and 1894 revolutions of Pallas = 737 of Jupiter. A new value for Jupiter's mass = 1/1042.86. Then follows (1816-1817) the computation of perturbation tables due to Jupiter, Saturn and Mars. In this latter work, Gauss was assisted by Encke and Nicolai.

In Astronomisches Jahrbuch 1816, page 234, Bode gives the best set of elements by Gauss up to that time (Elements G).

About 1824, Encke[7] used Gauss' elements based on early oppositions and computed the perturbations due to Jupiter. He reports that Gauss' elements with Jupiter's perturbations represent the opposition of 1823 as follows:

1823	$\Delta\alpha$	$\Delta\delta$
Oct. 9	$+13'.2$	$+25'.6$

He then gives Elements (H) for the epoch 1826, and with these computes the next ephemeris.

For the opposition in 1825, Encke[8] reports that the correction to the ephemeris is very large. But if the perturbations are included, the difference between observation and computation is as follows:

1825	$\Delta\alpha$	$\Delta\delta$
March 23	$+42''.6$	$-33''.2$

He then gives a set of elements for the epoch 1827, and computes an ephemeris for 1827.

By 1834 Encke[9] reports a deviation of Pallas from computed places amounting to 5′. He states this may be due to use of Laplace's value for Jupiter's mass. It will be necessary to recompute elements covering all observations.

In A. N. No. 636, Encke publishes osculating Elements I for each year from 1831 to 1838; his fundamental starting elements are for the epoch of 1810, January 0. In B. J. 1838, p. 286, Encke draws attention to an error which he had committed in neglecting the corrections for the secular variation of the obliquity. In the British Nautical Almanac for 1837 Airy points out this error to which Encke refers.

Galle[10] undertook the reinvestigation of the orbit based on oppositions 1816, 1821, 1827, 1830, 1834, 1836, making use of Airy's value for the mass of Jupiter 1/1048.69. The former perturbations were retained except for the change due to Jupiter's mass. The resulting Elements J represent the heliocentric longitude and latitude as follows:

	1816	1821	1827	1830	1834	1836
ΔL	—19″	+34″	+4″	—14″	+5″	—13″
ΔB	— 1	+ 4	+6	—10	+1	+ 4

Galle states these differences may be accounted for if the perturbations of Saturn and Mars were taken into account.

In A. N. No. 636 osculating Elements K are published for each year from 1839 to 1850. These were computed by Galle. The starting elements are those for epoch 1810, January 0. In computing the special perturbations, Encke and Galle used mass of Jupiter 1/1053.924.

From 1851 to 1870 Galle[11] continues the special perturbations by Jupiter and later with the elements of Günther also those by Saturn. These were used in computing the ephemerides published in the Astronomiches Jahrbuch from 1862 to 1870. (See Elements L.)

Beginning with the year 1871 and continuing to 1919, the Jahrbuch published and used Farley's[12] osculating elements for computing the ephemeris. (See Elements M and N). Farley's computation includes the perturbations by Venus, Earth, Mars, Jupiter and Saturn. His computations are also the basis for the ephemerides published in the British Nautical Almanac. With Farley's elements we have the following comparisons:

Corrections to Ephemerides.

	1883	1892	1895	1906	1908	1914
$\Delta\alpha$	$-1^{s}.4$	$-1^{s}.2$	$-1^{s}.0$	$-2^{s}.5$	$-5^{s}.4$	$-2^{s}.1$
$\Delta\delta$	$+2''.7$	$+0''.7$	$+0''.8$	$+8''.2$	-14.0	$+4''.3$

In Annales de l'Observatoire de Paris, Vol. I, Le Verrier publishes the results of his investigation on "Développement de la fonction perturbatrice relative à l'action de Jupiter sur Pallas. Calcul du terme dont dépend une inégalité à longue période du mouvement de cette dernière planète." Le Verrier states that the aphelion of Pallas is 54° from the intersection of the orbit with Jupiter. Consequently when Pallas is at aphelion the distance from Jupiter is increased on account of the great inclination of the orbit. This large inclination diminishes the effect due to the large eccentricity. Le Verrier gives the series for the reciprocal of the distance in a more convergent form and develops the equation in longitude depending on the argument 18 ♃ — 7 Pallas. The maximum of the term is 895″. In his report before the Paris Academy,[13] Cauchy compares his theory with the results by Le Verrier; his value for the inequality is 906″. Cauchy's investigation is more fully elaborated by M. Puiseux in Annales de l'Observatoire de Paris, Vol. VII. In Vol. VIII, *ibid.*, Hoüel has recomputed the inequality.

The development of the reciprocal of the distance was later extended by Tisserand.[14] He shows that the development depending upon the inclination and eccentricity is divergent in some parts of the orbit of Pallas and proceeds to give the analytical development and to apply it to the case of Pallas.

In Bulletin Astronomique Vol. XII, 1895, M. P. Bruck has published the results of his work on "The secular variations of the elliptic elements of Pallas due to the action of Jupiter." He used the method developed by Gauss and extended by Hill and Callandreau. He utilizes elements by Farley for the epoch 1878.

In 1910 George Struve[15] published his results on "Die Darstellung der Pallasbahn durch die Gauss'sche Theorie für den Zeitraum 1803 bis 1910." The result of his work based on 63 normal places is a more accurate value for the mean motion of Pallas (769″.1385). The new value for the annual motion of Pallas compared with Jupiter becomes $18n' - 7n = 123''$. The deviation between observation and computation still amounts to $\pm 4'$ which is attributed to the second order perturbations. These residuals are somewhat reduced by empirical terms.

In A. N. No. 205, p. 225, M. Viljev has published his "Recherches sur le mouvement de Pallas." He attempts to reduce the residuals from Struve's work ($\pm 4'$) by taking into account second order terms in the general perturbations, employing the method by Hill. He reports his results as negative.

REFERENCES

[1] Bode's A. J. 1805, p. 102.
[2] Bode's A. J. 1805, pp. 106, 111, 228.
[3] Bode's A. J. 1805, pp. 181, 182.
[4] Bode's A. J. 1806, pp. 179–180.
[5] Gauss Werke. vol. vi, pp. 3–24.
[6] Gauss Werke. vol. vii, pp. 413–610.
[7] Bode's A. J. 1828, pp. 154, 157.
[8] Bode's A. J. 1829, p. 158.
[9] B. J. 1837, pp. 249–250.
[10] B. J. 1839, pp. 237–238. A. N. vol. 14, p. 329.
[11] B. J. 1851, pp. 547 and 549. A. N. vol. 55, p. 194.
[12] British Nautical Almanac 1860 (not available here).
[13] Tisserand Méchanique Céleste, vol. iv, p. 278.
[14] Annales de l'Observatoire de Paris vol. xv.
[15] Dissertation Berlin. Ebernig, 1910 (not available here).

Additional references:
A. J. B., 1910, p. 188.
B. A. 28, p. 184.

TABLE 2.—*Elements—(2) Pallas*

Letter	Epoch	M.T.	L.	π	Ω	i	φ	μ	Authority	Equinox	Remarks
			° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
A.....	1802 March 31.	Seeberg	162 55 6.8	121 38 42	172 26 31	34 36 59	14 06 58	769.7263	Gauss	....	Gauss' Vth set Elements.
B.....	1802 Apr. 0....	....	162 51 14.5	122 03 02	172 28 57	34 50 40	14 15 31	760.98	Burkhardt	....	
C.....	1803.0........	Seeberg	221 28 54	121 24 13	172 28 08	34 38 20	14 13 06	769.416	Gauss	....	Gauss' VIth set of Elements.
D....	1803.0........	Göttingen	221 34 54	121 08 09	172 28 12	34 37 28	14 10 00	770.5010	Gauss	....	Based on oppositions 1803 to 1809 incl.
E.....	1803 156d.....	Göttingen....	254 56 38	121 07 54	172 28 21	34 37 44	14 12 46	770.7360	Gauss	....	Special perturbations
F.....	1810.0........	Göttingen....	47 23 05	121 08 05	172 31 44	34 36 09	14 02 50	769.1507	Gauss	....	Mean elements.
G....	1812 June 10...	Göttingen....	239 04 46	121 00 48	172 32 44	34 34 55	13 50 02	768.5746	Gauss	....	
H....	1826 June 24.5.	Paris........	254 21 34	120 58 08	172 37 10	34 35 56	14 06 13	769.4911	Gauss-Encke	June 24	No statement regarding elements by Encke.
I.....	1838 June 21...	Berlin........	109 14 25	121 45 35	172 38 31	34 38 31	13 49 49	768.7126	Encke	Epoch	Last set of osculating elements by Encke.
J.....	1810 Jan. 0....	Göttingen....	49 05 14	121 14 08	172 33 37	34 37 43	14 12 51	770.7388	Galle	Epoch	
K....	1850 Aug. 23...	Berlin........	338 52 59	121 21 48	172 44 00	34 37 33	13 52 32	768.4508	Galle	Epoch	Last set of Galle's osculating elements in A. N. 636.
L.....	1830 Jan. 0.0..	Berlin........	169 06 05	121 00 18	172 39 05	34 35 47	14 03 22	770.7516	Galle-Günther	1830	A. N. vol. 55, p. 194.
M....	1869 Sept. 25..	Berlin........	29 31 25	121 44 47	172 46 08	34 42 19	13 51 04	769.5682	Farley	Epoch	First set published by B. A. J. 1871.
N....	1913 May 5....	Berlin........	193 37 07	121 57 36	172 56 48	34 42 02	13 46 38	769.2236	Farley	Epoch	Published in B. A. J. 1917.

(3) JUNO

Juno was discovered by Harding at Lilienthal near Bremen, September 1, 1804. Gauss computed several orbits successively correcting the elements by new observations. Elements VII[1] corrected with Bessel's observations, 1807. Ephemeris for 1808, April-December, approximately given. (Elements A.) A number of additional orbits were computed by Gauss' students at Göttingen (Wachter, Möbius, etc.).

Wachter[2]: Elements, (eccentricity omitted) from the last four oppositions after Gauss' Method, (Neue Comment. der Göttingen K. Societät, Bd. 1) including the opposition 1812. Eccentricity supplied from Bode's Astronomische Jahrbuch, 1816, p. 233. (Elements B.)

Möbius[3]: Oppositions used: 1810, 1811, 1812, 1813. Correcting mean longitude by +4′55″, the representation of the observations 1815, March, is +8″ in longitude, and —51″ in latitude. (Elements C.)

Nicolai[4] at Seeberg, near Gotha, compared Gauss' observations with the orbit of Möbius, (empirically correcting L), determined the oppositions and derived new elements. Oppositions used: 1811, 1812, 1813, 1815. "Juno is nearly in conjunction with Jupiter and the perturbations may be large." (Elements D.)

Taking up the determination of the large perturbations by Jupiter by the method of special perturbations, Nicolai[5] derived a new set of elements. Oppositions used: 1811, 1812, 1813, 1815, 1816, 1817, 1818. Representation of observations in 1819, $\Delta\alpha + 2'.6$ $\Delta\delta$ $-0'.2$. (Elements E.)

Not satisfied with the representation of the observations by his last set of elements, Nicolai[6] extended the computation and determined new elements, which represented the observations of the "Atom" well in 1820. Oppositions used: 1805-1819. Special perturbations by Jupiter. Representation of the observations 1820, May, $\Delta\alpha$ —7″, $\Delta\delta$ —2″. (Elements F.) This computation of the special perturbations was continued for some years.

In 1823, Nicolai[7] derived his final set of elements, including the determination of the Jupiter mass, for which he found 1/1053.924, in agreement with the value Gauss had found from his theory of the motion of Pallas. The representation of the observations cannot be improved by taking Saturn or Mars into consideration, but Nicolai considers the possibility of the active mass of Jupiter changing with the body acted upon. From these elements osculating elements for 1826 were computed, taking account of the special perturbations by

Jupiter. Berliner Jahrbuch uses the elements by Nicolai until 1830. Fifteen oppositions used: 1804-1823. Special perturbations by Jupiter, (Saturn, Mars, negligible). Residuals in longitude —23″ to +27″, still show a run with the period of Jupiter. (Elements G.)

In 1832 new elements by Encke[8] were introduced, and are carried forward with special perturbations to 1865 by Bremiker and Powalky, for the ephemerides published 1832-1865. Perturbations by Jupiter with mass, 1/1053.924. (Elements H.)

Damoiseau[9] has published general perturbations in the Connaissance des Temps.

Hind[10] took over the work started by Nicolai, Encke, and Bremiker, to compute osculating elements for each opposition by special perturbations. The ephemerides are published in the Nautical Almanac. and the Berliner Jahrbuch. As a basis for this work he derived new elements. (Elements I.)

An attempt to apply Hansen's method of determination of the general perturbations was made by Berkiewicz,[11] starting with Hind's elements. The perturbations of the first order with regard to Jupiter, Mars, and Saturn were determined, also the constants of integration leading to a mean motion, 814″.090. No comparison with the observations is attempted.

Being aware that the corrections to the ephemerides computed according to Hind had increased to 3′ in 1887, Downing[12] undertook to correct Hind's elements. The errors of the tabular heliocentric places published in Greenwich Observations, 1864-1887, are discussed. Equations for the longitude and latitude corrections were set up expressed in terms of corrections to the elements and combined to eliminate the corrections to the radius vector. The mean motion is included in the solution and receives by far the greatest weight. The representation of the oppositions show a pronounced run in $\Delta\alpha$. These elements were used for the computation of the annual ephemerides to 1913 in the Nautical Almanac and to 1917 in the Berliner Jahrbuch. (Elements J). Oppositions used: 1864-1887. Special perturbations by Venus, the Earth, Mars, Jupiter, Saturn. Representation varies from —3″ to +4″ in $\Delta\alpha \cos\delta$, and —1″ to +2″ in $\Delta\delta$. Large residual (—11″) for 1874. Representation for 1890 is then in α+3″, in $\delta\pm$0″ against —65″ and —6″, according to Hind's computations.

Since 1917 Ephemeriden der Kleinen Planeten gives mean elements by Boda[13] derived by the method of Brendel. (Elements K.) Mean elements. Perturbations by Jupiter according to Brendel, A. N., Vol. 195, p. 417. Expected representation ±0°.5 to year 2000. Oppositions not stated.

The absolute perturbations according to the method of Hansen (?), have been computed by Viljev.[14]

REFERENCES

[1] Bode's B. J. 1811, p. 136. Gauss Werke Bd. 6.
[2] Bode's B. J. 1815, p. 248.
[3] Bode's B. J. 1817, p. 213.
[4] Bode's B. J. 1818, p. 264.
[5] Bode's B. J. 1820, p. 200.
[6] Bode's B. J. 1822, p. 218.
[7] Bode's B. J. 1826, p. 224.
[8] A. N. Bd. 27, p. 177.
[9] Connaissance des Temps, 1846. Additions. Not available here.
[10] Nautical Almanac, 1859. Appendix. Not available here.
[11] A. N. vol. 72, p. 1, p. 145, p. 289.
[12] N. M. vol. 50, p. 487.
[13] A. N. vol. 200, p. 1.
[14] Bulletin-Soc. Astr. Russ. vol. 22. Not available here.

Table 3.—*Elements—(3) Juno*

Letter	Epoch	M. T.	M.	π	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A.....	1805.............	Göttingen.......	42 37 3.7	53 19 0.2	171 4 28.2	13 4 26.2
B.....	1811.............	Göttingen.......	177 48 21.0	53 15 10.1	171 9 16.7	13 4 17.2
C.....	1810.............	Göttingen.......	95 29 53.2	53 6 43.0	171 6 45.0	13 4 12.9
D....	1815 Dec. 31.0....	Göttingen.......	230 11 34.2	53 14 53.8	171 9 58.9	13 4 0.1
E.....	1819.............	Mannheim......	117 45 2.84	53 32 56.09	171 6 50.23	13 3 37.29
F.....	1820 May 11......	Mannheim.......	230 9 22.03	53 31 6.52	171 8 11.08	13 3 47.20
G....	1810.............	Göttingen.......	95 25 9.82	52 58 35.80	171 6 28.52	13 4 18.99
H....	1826 Nov. 1......	Berlin..........	351 43 27.3	53 11 13.4	170 56 57.4	13 3 28.4
I.....	1861 Nov. 21.0...	Greenwich......	58 34 1.0	54 9 3.3	170 59 49.7	13 2 58.8
J.....	1861 Nov. 21.0...	Greenwich......	58 34 1.83	54 9 3.82	170 54 45.69	13 2 58.45
K....	1900 Jan. 0.......	....	324 12	55 36	170 42	13 2

Letter	Epoch	M. T.	φ	μ	Equinox	Authority
			° ′ ″	″		
A.....	1805...........	Göttingen.......	14 48 11.5	813.8468	....	Gauss VII
B.....	1811...........	Göttingen.......	14 44 12	813.25748	1811	Wachter
C.....	1810...........	Göttingen.......	14 43 9.5	812.7140	1810	Möbius
D....	1815 Dec. 31.0..	Göttingen.......	14 43 28.84	812.9304	1816.0	Nicolai
E.....	1819...........	Mannheim......	14 53 17.44	813.86981	1819	Nicolai
F.....	1820 May 11....	Mannheim......	14 55 1.78	814.40238	1820 May 11	Nicolai
G....	1810...........	Göttingen.......	14 44 39.19	813.4837354	1810	Nicolai
H....	1826 Nov. 1....	Berlin...........	14 53 22.6	813.88514	1810	Encke
I.....	1861 Nov. 21.0..	Greenwich.......	14 47 14.1	813.34555	1861 Nov. 21.0	Hind
J.....	1861 Nov. 21.0..	Greenwich.......	14 47 13.81	813.35271	1861 Nov. 21.0	Downing
K....	1900 Jan. 0.....	....	14 50	813.434	1900	Boda

(4) VESTA

Vesta was discovered by Olbers[1] at Bremen on March 29, 1807.

Preliminary elements were computed by Gauss[2] and also by Burkhardt.[3] The third set of Elements A by Gauss is based on observations from March 29 to July 11. They were used to compute the ephemeris for 1808-1809.

The preliminary work of Gauss was continued by Gerling,[4] who supplied the ephemeris for a number of years. His last set of Elements B, are based on the first six oppositions.

Burkhardt's preliminary work was continued by Daussy[5] (reference not available here). In his work he took into consideration the perturbations by Jupiter, Saturn, and Mars, and was able to represent the first seven oppositions satisfactorily. On account of the small eccentricity and inclination, the methods of La Place and Le Verrier were sufficient.

On account of increased error in the ephemeris for 1818 based on Gerling's first elements, Encke[6] computes a set of Elements C, based on oppositions 1812, 1815, 1816, and 1818.

In Astronomisches Jahrbuch 1829, pp. 156-158, Encke gives the results of his work on Vesta based on fourteen oppositions by also using Nicolai's value of Jupiter's mass (1/1054), for the perturbations due to Jupiter. He also makes use of Daussy's tables for the perturbations of Saturn and Mars. His new set of Elements D represents the oppositions from 1807 to 1825 within $\pm 6''$. Elements D are then brought up to the epoch of 1827. Astronomisches Jahrbuch from 1830 to 1866, contains the elements and ephemerides computed by Encke. (See Elements E.) The method of computation[7] was that of the variation of the elements by special perturbations of Jupiter. In this work Encke was assisted by Bruhns and Schiaparelli. In A. N. 332, Encke comments on the poor results obtained for planets (1), (2) and (3), by using Laplace's value for the mass of Jupiter and also publishes a new value of Jupiter's mass (1/1050), obtained from the results of Vesta.

The Berliner Jahrbuch for 1868 publishes mean Elements F by Brünnow.[8] They are also published in Watson's Theoretical Astronomy. Brünnow completes the work of Wolfers and Galle[9] who developed expressions for the perturbations in longitude and radius vector after Hansen's method. Brünnow's elements represent the oppositions from 1810 to 1851 within $-6''$ to $+10''$. For Jupiter's mass he used 1/1050.

From 1871 to 1910 the Berliner Jahrbuch publishes elements (see Elements G), by Farley.[10] His work is based on twelve oppositions, 1840 to 1855. This work forms the basis for later investigations of the general perturbations.

Probably the most extensive work on a minor planet are the tables of Vesta by Leveau published in Annales de l'Observatoire de Paris Mémoires, XV, XVII, XX, XXII, XXV. The method applied by Leveau is that of Hansen "Auseinandersetzung einer Zweckmässigen Methode zur Berechnung der Absoluten Störungen der Kleinen Planeten, I, II, III." The explanation for the choice of this method is that the application to Vesta is a preparatory study to the motion of Pallas, as Gauss' theory of Ceres was a preliminary study to his theory of Pallas. Memoir XV contains the perturbations of the first order of the masses of Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune. The memoir concludes with the determination of the constants of integration and the expressions for $n\delta z$, ν, u seci and a representation of an observation 1858, April, 23.5 as follows: $\Delta\alpha = -0^s.1$ and $\Delta\delta = -0''.4$. Memoir XVII contains the terms depending upon the square of the mass of Jupiter in $n\delta z$ and 2ν. The effect on u seci becomes noticeable after 100 years. Memoir XX contains the terms depending upon the product of the masses and concludes with a set of mean elements and corresponding expressions for $n\delta z$, ν, and u seci. The mean elements given in memoir XX are slightly changed in memoir XXII (see Elements H), on account of some perturbations of the second order that Farley had included and which form the basis of Leveau's work. Memoir XXII contains the comparisons with 215 normal places founded on 5000 observations extending from 1807 to 1889. The computation has been changed to conform to the solar tables by Newcomb. The correction to the mean motion is zero. The new Elements I represent the observations in right ascension between $-2''$ and $+4''$ and in declination between $-1''$ and $+4''$. A new value of Jupiter 1/1046, and Mars 1/3648000. The perturbations are collected in tables introducing the mean anomaly and corrective terms for the eccentric anomaly. Memoir XXV contains some supplementary terms depending upon the product of the masses. One of the larger terms has a period of 3000 years. The effect of the critical terms, which are mentioned in the beginning of his work, shows itself by comparing these terms before and after integration. The coefficients of corresponding terms are ten or twenty times larger, whereas the other terms mostly decrease.

Leveau has made a comparison between observations and calculation of later positions[11] showing the residuals as obtained from the British Nautical Almanac, and his own computations. The following comparisons are illustrations of the results:

		1890	1892	1894	1896	1898
Nautical Almanac	$\Delta\alpha$	$+1^s.18$	$+1^s.00$	$+1^s.73$	$+1^s.71$	$+2^s.91$
	$\Delta\delta$	$+0''.9$	$-5''.8$	$+8''.9$	$-2''.0$	$+16''.3$
Leveau	$\Delta\alpha$	$+0^s.03$	$+0^s.01$	$+0^s.25$	$+0^s.06$	$+0^s.19$
	$\Delta\delta$	$+0''.4$	$+0''.5$	$+2''.0$	$+1''.5$	$+1''.0$

Further results of Leveau's theory are published in Comptes Rendus. T. 145, p. 903-906, "Détermination des Éléments Solaires et des Masses de Mars et de Jupiter par les Observations Méridiennes de Vesta." Extending the comparison with the meridian observations from 1807 to 1904 and taking into consideration the masses of Jupiter and Mars and also the solar elements, Leveau determines a new set of smaller corrections to the elements of Vesta and for Jupiter's mass, 1/1046, and mass of Mars 1/3601280. The tables of residuals shows the poor quality of the meridian observations before 1826. A period of 36 years, (three revolutions of Jupiter, or ten of Vesta), points to the effect of the critical terms in the residuals; the amplitude is about 1″. The effect of the earlier observations on the value for the mean motion is also illustrated by the last residuals.

In Annales de l'Observatoire Astronomique de Toulouse, T. I., B. 1 to B. 90, M. J. Perrotin published his extensive investigation on the "Théorie de Vesta," applying the method of Le Verrier (Annales de l'Obs. de Paris, T. X.). The method consists first of deriving mean elements from previous osculating elements by computing provisional periodic perturbations and applying these to the osculating elements for a first approximation. In order to avoid considerable labor, Perrotin starts with a certain fixed major axis and develops corrective terms for the variation in the assumed value. The final mean motion is determined from two extreme groups of observations in 1807 and 1876 when the planet was near the same place in its orbit. The perturbative function is developed by Le Verrier to the seventh degree in the inclination and eccentricity; thus Perrotin includes terms of $10n'-3n$. Venus, Earth, Mars, Jupiter, and Saturn are taken into account. Derived from the secular terms e is always smaller than 0.15, the mean motion of the perihelion is $+38''$, that of the node $-38''$, and the inclination remains less than 9°. The terms of the second order are then considered. Those due to the square of Jupiter's mass of the second degree are small. Those depending on the product of the masses are more important, especially those depending upon $5n''-2n'$,

$-2n''+4n'-n$, $2n''+9n'-3n$. For getting these terms the development of the perturbative function is used to the seventh degree and for determining the terms of the eighth degree the method of Cauchy, as extended by Puiseux, is used. No comparison with observations is attempted.

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TABLE 4.—*Elements—(4) Vesta*

Letter	Epoch	L	M	ω	Ω	i	φ	μ	Authority	Remarks
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″		
A.....	1807 March 31, Bremen.	192 23 30	....	146 32 04	103 18 28	7 08 11	4 54 18	981.7087	Gauss	3rd set of preliminary orbits.
B.....	1816.0, Göttingen......	341 07 43	....	146 29 06	103 13 29	7 08 10	5 07 56	977.8933	Gerling	Based on first six oppositions.
C.....	1818.0, Seeberg	179 38 30	....	146 20 06	103 11 38	7 08 09	5 06 24	977.7020	Encke	
D....	1810.0, Paris..........	105 53 16	216 04 49	146 40 06	103 08 20	7 08 12	5 09 39	978.2967	Encke	
E.....	1831 July 23, Berlin....	....	195 35 26	145 51 09	103 20 28	7 07 57	5 04 51	977.7554	Encke	B. J. 1831, p. 249.
F.....	1810 Jan. 0, Berlin.....	....	216 42 26	146 08 07	103 11 22	7 08 05	5 05 36	977.6339	Brünnow	
G....	1869 May 13, Berlin....	....	345 13 50	146 38 57	103 26 23	7 07 53	5 07 42	977.7104	Farley	B. J. 1871.
H....	1857 Jan. 1.0, Paris....	....	198 20 17	147 10 59	103 23 19	7 08 07	5 06 05	977.6324	Leveau	Mem. XXII, p. A1.
I.....	1857 Jan. 1.0, Paris....	....	198 20 33	147 10 40	103 23 20	7 08 06	5 06 04	977.6325	Leveau	Mem. XXII, p. A57.

(10) HYGIEA

De Gasparis at Naples announced to the General-Secretary of Corrispondenza Scientifica that he had discovered this planet April 12, 1849. The magnitude was 9–10.[1]

A large number of preliminary orbits were computed by Encke,[2] Luther,[3] Brorsen,[4] Quirling,[5] d'Arrest,[6] Hensel,[7] Santini,[8] from observations in the first opposition with mean motions ranging from 601″ to 652″. The first more certain, Elements A, were those by d'Arrest[9] from observations in 1849 and 1850. These elements were published in B. J. 1853–55, brought forward with special perturbations by Jupiter.

In B. J. for 1856 two sets of elements were published: one by Chevallier at Durham, B, from observations in 1851 and 1852; and one, C, by Zech at Tübingen. The latter (corrected for an error[10] which did not affect the ephemeris) were based upon four oppositions and represented the opposition 1854 by $+0^s.5$ in α and $-1''$ in δ. This was the beginning of Zech's important work which made this planet suitable for testing theories in the case of near commensurability.

In B. J. 1858, new Elements D, by Zech,[11] are published, based upon five oppositions taking the perturbations by Jupiter, Saturn, and Mars into account. These elements are very closely the same as those v. Zeipel selected from Zech's manuscript. In this the general perturbations by Jupiter, Saturn, and Mars were computed and partly tabulated. The basic elements for this computation were founded on eight oppositions. After Zech's death in 1864 his computations of the general perturbations of the planets (5) and (10) were discontinued by the Recheninstitut, whereas his elements and special perturbations for (10) Hygiea were used in B. J. until 1875 by Powalky and Becker. In 1873 the correction to the ephemeris had increased to -4^s in α and $-20''$ in δ.

In B. J. 1876 E. Becker published a preliminary set of elements which finally was corrected to the Elements E, given by Bauschinger[12] based upon the oppositions 1868, 1869, 1871, 1873 and 1874. Special perturbations by Jupiter and Saturn were included and brought forward to the osculation 1898, December 20.

The elements given in Kleine Planeten for 1920 are Becker's brought forward by Strehlow with Jupiter's perturbations. The mean motion is corrected empirically by $+0''.07$ (since 1898, August 22), representing the fourteen oppositions since 1900 within $\pm 0^m.2$. Osculation 1920 January 0. Elements F.

In the mean time v. Zeipel[13] had computed the tables for the general perturbations by Jupiter according to the method by Bohlin in the case of near commensurability, Hecuba group. As a test he applied his tables to the orbit of (10) Hygiea and started with the Elements G (a) from Zech's manuscript. These were first transformed to mean elements and then compared with nine oppositions from 1849–1884 with the aid of his tables. A least squares solution gave the corrected Elements H.

The work of computing tables after Bohlin's method for the Hecuba group had also been undertaken by A. O. Leuschner[14] and an application to (10) Hygiea was made by Miss E. Glancy and Miss S. H. Levy[14]. In order to compare the results with those of v. Zeipel, Miss Glancy[15] used the Berkeley tables and after a comparison with nine oppositions between 1849–1884 obtained the Elements I, starting with the mean Elements G(b). Further comparisons were made by representing observations in 1910, 1914, 1917. The residuals before solution from Zech's elements with the Berkeley tables were —11′ to +10′ in the plane; from Zech's elements and v. Zeipel's tables —24′ to +5′. After Miss Glancy's solution the residuals are —8′ to +7′, and after v. Zeipel's solution —7′ to +8′; but in 1917 they are +9′ and +19′ respectively, apparently in favor of the Elements I and the Berkeley tables.

The residuals from Zech's elements and the Berkeley tables (—11′ to +10′) show a decided periodicity of 30 years, thus pointing to the influence of Saturn. Later observations in 1917 and 1921 are represented much better (0′ and +10′) by the Berkeley tables and Zech's elements than by any of the solutions.[16] The best future representation may be expected from Zech's elements G(b) and the Berkeley tables. The residuals, probably chiefly due to Saturn, keep within fixed limits ±10′. The most obvious next step would be to correct the residuals for some of the earlier oppositions by means of the perturbations of Saturn and Mars, available in manuscript in the Recheninstitut. Until that shall have been done, no corrections should be applied to Zech's elements, which appear to be the best available.

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TABLE 5.—*Elements—(10) Hygiea*

Letter	Date	M.T.	M	π	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A.....	1849 Apr. 15.0..	Berlin......	330 52 8.56	227 49 54.23	287 37 8.64	3 47 15.51
B.....	1851 Sept. 28.5..	Berlin......	128 44 20.7	228 2 32.1	287 30 8.3	3 47 8.3
C.....	1851 Sept. 17.0..	Berlin......	126 59 37.2	227 48 9.2	287 38 37.6	3 47 9.22
D.....	1851 Sept. 17.0..	Berlin......	126 59 48.76	227 47 58.77	287 38 34.21	3 47 9.29
				ω		
E.....	1874 Dec. 26.0..	Berlin......	174.55.30.0	312 40 30.5	285 18 57.5	3 47 43.2
F.....	1925 Jan. 0.0..	Greenwich..	181 38 49.2	305 25 22.8	285 52 55.2	3 48 46.8
				π		
G (a)..	1851 Sept. 17.0..	Berlin......	126 59 48.6	227 46 36.6	287 37 11.4	3 47 8.4
G (b)..	1851 Sept. 17.0..	Berlin......	121 51 58	230 47 49.6	287 37 11.28	3 47 8.5
H.....	1851 Sept. 17.0..	Berlin......	121 35 27.6	231 2 9	287 8 33.6	3 47 45.6
I......	1851 Sept. 17.0..	Berlin......	121 31 53.8	231 4 56.6	287 27 28.1	3 47 30.1

Letter	Date	M.T.	φ	μ	Equinox	Authority
			° ′ ″	″		
A.....	1849 Apr. 15.0.....	Berlin....	5 47 55.37	634.6406	1849.0	d'Arrest VI
B.....	1851 Sept. 28.5.....	Berlin....	5 46 34.7	634.83504	....	Chevallier
C.....	1851 Sept. 17.0.....	Berlin....	5 46 16.8	634.84564	1851 Sept. 17	Zech
D.....	1851 Sept. 17.0.....	Berlin....	5 46 16.57	634.84912	1851 Sept. 17	Zech
E.....	1874 Dec. 26.0.....	Berlin....	6 18 23.7	636.58673	1870.0	E. Becker
F.....	1925 Jan. 0.0.....	Greenwich	6 40 48.0	638.517	1925.0	Strehlow
G (a)..	1851 Sept. 17.0.....	Berlin....	5 46 16.8	634.850	1850.0	Zech
G (b)..	1851 Sept. 17.0.....	Berlin....	6 23 8.9	636.8566	1850.0	Zech
H.....	1851 Sept. 17.0.....	Berlin....	6 22 1.2	636 849	1850.0	v. Zeipel
I......	1851 Sept. 17.0.....	Berlin....	6 21 31.0	636.86105	1850.0	Glancy

(28) BELLONA

Discovered by R. Luther[1] at Bilk near Düsseldorf, March 1, 1854.

Preliminary elements were published by Bruhns,[2 3 4] Chevallier[5] and Ruemker,[5] Oudemans.[6]

From 141 observations formed into five normal places Bruhns[7 8] derived, originally using four longitudes and two latitudes corresponding to the first, second, fourth and fifth normals, the first reliable Elements A. As Elements A differ considerably from his previous elements and those of Oudemans. He made a comparison of ephemerides which satisfied him regarding the correctness of Elements A. This case is somewhat indeterminate and small errors of observation would produce considerable changes in resulting elements. From Elements A Bruhns[7] has published an ephemeris for 1855.

Further ephemerides are published by Bruhns, among them for 1856,[9] 1866,[10] (a star correction[11] November 29 by Engelmann), for 1867,[12] correction $\Delta\alpha$ —50^s, $\Delta\delta$ —$3'.4$ by Tietjen,[13] and for 1871–72.[14]

Other elements by Bruhns are published in the B. J. from 1857 to 1860.

The B. J. from 1861 to 1891 contains new elements and ephemerides by Bruhns originally based upon the observations of the first four oppositions with perturbations by Jupiter, Saturn, and Mars. Elements C.[19] Whether these elements are merely brought forward by perturbations or contain corrections is difficult to determine.

From 1892 the B. J. uses von der Groeben's elements instead of those by Bruhns.

By a process of successive correction of osculating elements and special perturbations by Jupiter, Saturn and Mars (perturbations by the Earth and Venus were found negligible), based on 16 normal places extending over a period of thirty-two years from 1854 to 1886, von der Groeben,[15] starting with a set of elements osculating for 1870, September 18, derived Elements Ba, Bb, Bc, Bd osculating for different epochs from 1861 to 1882. These elements were brought forward with special perturbations of Jupiter, Saturn and Mars to the epoch 1886, February 26, Elements Be, and to the epoch 1889, October 28, Elements Bf. Elements Bf are adopted by the B. J. for 1892[16] and 1893.[17] Three observations by Ball at Lüttich, October 31 to November 15, 1889, are well represented by the ephemeris. Mean $\Delta\alpha$ $+^s.19$, mean $\Delta\delta$ —$''.5$.

From seven observations at Washington and Tacubaya with star places newly determined by Bruns and four observations by himself at Düsseldorf, Luther[18] obtained a mean correction to the ephemeris from von der Groeben's Elements Bf brought forward by special perturbations of $\Delta\alpha$ $-0^s.42$, $\Delta\delta$ $+3''.4$ in 1894.

The Elements Bg[20] to Bu given in the B. J. and in Kleine Planeten from 1894 to 1919 are von der Groeben's elements probably brought up to date in the same manner as before with the special perturbations by Jupiter, Saturn and Mars and without any other correction.

The following is a partial list of published corrections to the B. J. ephemerides from von der Groeben's elements:

1902, August 7	$+\ 8^s$	$+\ 0'$	Luther[21]
1903, October 26	$+6^m38$	$+25'.2$	Luther[22]
1905, March 1	$+\ 2.77$	$-\ 8'.3$	Iwanowski[23]
1906, June 20	$+\ 8.99$	$+11.3$	Luther[24]
1907, September 7	$+\ 21.68$	$+1'\ 23''.1$	Luther[25]
1908, November 28	$-\ 17.77$	$+0'\ 9''.0$	Luther[26]
1916, August 8	$+0^m.3$	$+1'$	Luther[27]
1917, December 4	$+1^m.4$	$+7'$	Luther[28]
1919, April 1	$-1^m.8$	$+6'$	Luther[29]

In a dissertation on the Jupiter perturbations of the group of small planets whose mean daily motions are in the neighborhood of 750″, D. T. Wilson[30] gives an application of the Hansen-Bohlin method to the Jupiter perturbations of this group. "The integration divisors for certain values of the integers n, r and s become as small as 0.2. These terms increase rapidly as the series advance. They were computed to the third power of the eccentricities and to the fourth power of w. It was found that all the terms of the third and fourth powers of w and some of those of the third power of the eccentricities are negligible when the eccentricity of the disturbed planet does not exceed 0.34 and when the mean daily motion lies within the limits 720″ to 780″. Therefore only those terms of the third power of the eccentricities which are appreciable within the above limits have been retained. All the secular terms have been computed to the fourth power of eccentricities."

By means of these tables the Jupiter perturbations of Bellona were computed and compared with the results previously obtained by Hansen's method by Bohlin.[31]

The mass of Jupiter is taken as 1: 1048 in the tables by Wilson.

Of the three applications of these tables by D. T. Wilson that of Bellona is by far the most interesting. This depends on the greater proximity to the commensurability (748″) and the cross position of the line of apsides to that of Jupiter. The inequality 2g–5g′ in $\eta\delta z$ is the largest of all and amounts to 40′. But the difference between the coefficients computed by Hansen's and Bohlin's methods is large (2′) and the comparison with the observations ought to decide between the application of these methods to numerous planets in this group.

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TABLE 6.— *Elements—(28) Bellona*

Letter	Epoch	M. T.	M.	π	☊	i	φ	μ	Equinox	Authority
			° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″		
A..........	1854 Mar. 0.0.......	Berlin..........	36 43 21	122 18 29	144 43 7	9 22 33	8 53 54	767.5226	1854.0	Bruhns
				ω						
Ba..........	1861 Sept. 25.0.......	Berlin..........	264 55 34	338 10 21	144 39 21	9 21 26	8 38 11	765 9065	1860.0	v. d. Groeben
Bb..........	1866 Oct. 9.0.......	Berlin..........	296 28 59	338 20 32	144 38 29	9 21 38	8 47 4	766.8382	1870.0	v. d. Groeben
Bc..........	1870 Sept. 18.0.......	Berlin..........	243 24 36	338 6 45	144 37 17	9 21 47	8 50 59	767.3374	1870.0	v. d. Groeben
Bd..........	1882 Apr. 28.0.......	Berlin..........	64 43 25	339 18 14	144 37 35	9 21 31	8 34 10	765.8751	1880.0	v. d. Groeben
Be..........	1886 Feb. 26.0.......	Berlin..........	2 10 25	339 40 53	144 44 36	9 21 35	8 35 28	766.1202	1890.0	v. d. Groeben
Bf..........	1889 Oct. 28.0.......	Berlin..........	287 46 46	339 19 23	144 39 28	9 21 27	8 40 59	766.4180	1890.0	v. d. Groeben
Bg..........	1891 Feb. 20.0.......	Berlin..........	30 3 12	339 11 38	144 39 11	9 21 28	8 41 24	766.1111	1890.0	v. d. Groeben
Bh..........	1893 Nov. 16.0.......	Berlin..........	243 19 49	338 53 47	144 38 42	9 21 31	8 42 58	766.8942	1890.0	v. d. Groeben
Bi..........	1896 Apr. 14.0.......	Berlin..........	71 10 8	338 24 33	144 43 21	9 21 42	8 41 6	766.6272	1900.0	v. d. Groeben
Bj..........	1897 June 28.0.......	Berlin..........	164 45 50	338 30 4	144 43 20	9 21 42	8 40 6	766.2606	1900.0	v. d. Groeben
Bk..........	1898 Sept. 11.0.......	Berlin..........	258 21 44	338 30 59	144 43 16	9 21 37	8 38 55	765.9782	1900.0	v. d. Groeben
Bl..........	1900 Feb. 13.0.......	Berlin..........	8 58 37	338 31 49	144 42 22	9 21 38	8 38 00	766.2998	1900.0	v. d. Groeben
Bm..........	1903 Nov. 5.0.......	Berlin..........	295 51 7	340 41 44	144 34 37	9 23 12	8 38 1	766.7700	1900.0	v. d. Groeben
Bn..........	1905 Apr. 8.0.......	Berlin..........	46 31 26	340 48 4	144 42 37	9 23 4	8 39 10	766.4981	1910.0	v. d. Groeben
Bo..........	1906 June 22.0.......	Berlin..........	140 8 28	340 50 1	144 41 37	9 23 4	8 40 14	766.3312	1910.0	v. d. Groeben
Bp..........	1907 Sept. 5.0.......	Berlin..........	233 47 46	340 50 12	144 41 21	9 23 12	8 41 35	766.6662	1910.0	v. d. Groeben
Bq..........	1908 Dec. 28.0.......	Berlin..........	336 3 47	340 49 47	144 41 19	9 23 14	8 42 32	766.9646	1910.0	v. d. Groeben
Br..........	1910 Apr. 22.0.......	Berlin..........	78 25 24	340 38 24	144 40 13	9 23 1	8 45 7	766.6520	1910.0	v. d. Groeben
Bs..........	1911 July 6.0.......	Berlin..........	172 23 32	340 25 38	144 39 15	9 23 3	8 45 42	767.2816	1910.0	v. d. Groeben
Bt..........	1912 Oct. 28.0.......	Berlin..........	274 51 16	340 18 9	144 39 2	9 23 58	8 45 5	766.913	1910.0	v. d. Groeben
Bu..........	1925 Jan. 0.5.......	Greenwich......	142.319	340.308	144.854	9.398	8.751	766.913	1925.0	v. d. Groeben
				π						
C..........	1857 Dec. 15.0.......	Berlin..........	331 41 53	122 24 28	144 38 58	9 21 24	8 39 0	766.1418	1854.0	Bruhns.

(93) MINERVA

Discovered by J. Watson[1] 1867, August 24, at Ann Arbor, and observed on three successive days. Estimated to be of 11th magnitude.

P. Lehmann[2] in Berlin computed his first orbit from the observations to October 2, covering an heliocentric arc from 60° to 75° after perihelion. An ephemeris for 1867, November and December, deviated $+0^s.33$ in α, $-2''.0$ in δ from the observations. These elements were repeated in B. J. 1870. Elements A.

Lehmann changed his Elements A to the Elements B given in B. J. 1871 without statement of the observations used, the perturbations applied or the representation of the observations. The most prominent changes are the improvement of the longitude of perihelion by 1°, and also the increase of the eccentricity, besides diminution of the mean motion by 1″ a day, to a value as much below the mean mean motion as the osculating value was above. This error in the mean motion made a new computation necessary which Lehmann published in B. J. 1872 without explanation. Elements C. A further improvement was obtained by the next set of Elements D by Lehmann in B. J. 1873. These elements were (perhaps) brought up with perturbations to the new osculation in 1872 as published in B. J. 1874 and 1875. Elements E. To the elements M, μ and ϕ corrections were applied. The resulting Elements F are given in B. J. for 1876-1880. Then Lehmann undertook to make a final determination of the orbit from the first 7 oppositions with special perturbations by Jupiter and Saturn, (as stated in Bauschinger, Tabellen, etc.) These Elements G were probably published in B. J. with change of osculation until the issue 1913.

Berberich corrected the Elements G empirically by the observations in the oppositions 1899, 1902, 1907, 1908, 1911, and derived the Elements H, which were published in B. J. 1914. B. J. 1915 gives the elements by Leuschner.[6]

The general perturbations by Jupiter were developed by W. S. Eichelberger.[3] The basic elements were obtained "from a special discussion, by the author, of the observations from 1867 to 1879, inclusive." Elements I. The method is that of Hansen for absolute perturbations of the first order retaining the eccentric anomaly in the argument. The constants of integration were determined. With these perturbations and the preliminary Elements I the observations from 1867 to 1884 were compared. A least squares solution was made and the final Elements J obtained. The comparison with the observations is not given.

Observations by T. J. J. See in 1899 Sept. compared with a manuscript ephemeris by Eichelberger gave the corrections $+0^s.8$ in α, $+50''$ in δ. "The comparison would indicate that Eichelberger's theory is very good." Same year and date Coddington observed Minerva and compared "with an ephemeris furnished by Professor Newcomb." Same corrections as See found to Eichelberger's ephemeris.

The planet discovery 1902 HQ (February 25), by Wolf[4] was suspected to be (93) Minerva and the identity was confirmed by two observations from Bordeaux.[5]

The work of Eichelberger was undertaken originally under the auspices of the Watson Trustees, but the Trustees suspected that the small residuals which resulted from a comparison of theory and observation were due to some error. Investigation of the sources of the suspected error undertaken by Leuschner at the request of the Trustees confirmed Eichelberger's work, the representation of the observations being found entirely satisfactory, in view of the fact that only first order perturbations of Jupiter were considered.

Leuschner[6] revised the elements by including in the least squares solution further oppositions to 1902, so that his elements are based on oppositions extending from 1867 to 1902. The elements differ only slightly from those by Eichelberger. In the Perturbations and Tables of Twelve Watson Asteroids,[9] Eichelberger's perturbations are retained without change. Observations of recent years are well represented by the ephemerides published in Kleine Planeten by the Berlin Recheninstitut on the basis of these elements and tables, as is indicated by comparison with approximate photographic positions at Königstuhl[7] in 1918 and at Algiers in 1921.[8] Further corrections of the elements should be undertaken only on the basis of perturbations by Jupiter of the second order, and of perturbations by other major planets.

For the group of minor planets having a mean motion of about 750″ (the Minerva group), D. T. Wilson[9] has computed tables after the method of Bohlin. No application of this theory seems to have been made to (93).

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[2] A. N. vol. 70, p. 205.
[3] Memoirs of the National Academy of Sciences. vol. viii, third memoir. Washington, 1899.
[4] A. N. vol. 158, p. 95.
[5] A. N. vol. 158, p. 175; A. N. vol. 160, p. 352.
[6] Memoirs of the National Academy of Sciences. vol. x, seventh memoir. Washington, 1910.
[7] Eph. Z, A. N. 1918/558.
[8] C. O. M. 1921/171.
[9] Astronomiska Iakttagelser och Undersökningar. vol. 10, No. 1, Stockholm.

TABLE 7.—*Elements—(93) Minerva*

Letter	Epoch	M	π	☊	i
		° ′ ″	° ′ ″	° ′ ″	° ′ ″
A........	1867 Oct. 2.0....Berlin......	66 47 58.8	276 39 54.8	5 2 28.0	8 35 34.9
B........	1867 Oct. 2.0....Berlin......	67 1 59.1	275 38 16.3	5 4 11.4	8 36 31.8
C........	1870 May 1.0....Berlin......	270 51 42.1	275 2 55.0	5 4 16.2	8 36 17.6
D........	1870 May 1.0....Berlin......	270 53 55.4	275 0 36.8	5 4 26.8	8 36 34.6
E........	1872 Nov. 6.0....Berlin......	109 32 42.8	274 43 34.4	5 3 40.3	8 36 34.3
F........	1872 Nov. 6.0....Berlin......	109 32 48.4	274 43 34.4	5 3 40.3	8 36 34.3
G........	1879 Feb. 3.0....Berlin......	241 7 28.6	274 42 35.8	5 9 11.9	8 36 3.2
H........	1911 Jan. 31.5...Berlin.....	236 37 30	277 36 31.2	5 4 31.2	8 35 28.0
I........	1872 Nov. 2.0....Greenwich..	108 37 48.4	274 47 41.4	5 5 25.0	8 36 21.6
J........	1872 Nov. 2.0....Greenwich..	108 28 35.7	274 49 19.2	5 5 17.5	8 36 23.6
K........	1875 Jan. 0.0....Greenwich..	278 32 8	274 51 41	5 7 8	8 36 20

Letter	Epoch	φ	μ	Equinox	Computer
		° ′ ″	″		
A......	1867 Oct. 2.0....Berlin............	7 39 29.5	776.43667	1867.0	Lehmann
B......	1867 Oct. 2.0....Berlin............	8 3 47.9	775.5500	1870.0	Lehmann
C......	1870 May 1.0....Berlin............	8 3 55.5	776.41806	1870.0	Lehmann
D.....	1870 May 1.0....Berlin............	8 4 38.9	776.51030	1870.0	Lehmann
E......	1872 Nov. 6.0....Berlin............	8 4 43.5	776.47953	1870.0	Lehmann
F......	1872 Nov. 6.0....Berlin............	8 4 45.1	776.49465	1870.0	Lehmann
G......	1879 Feb. 3.0....Berlin............	7 59 4.8	775.63887	1880.0	Lehmann
H.....	1911 Jan. 31.5...Berlin............	8 15 30	775.6316	1910.0	Berberich
I......	1872 Nov. 2.0....Greenwich........	8 5 0.5	776.51130	1872.0	Eichelberger
J......	1872 Nov. 2.0....Greenwich........	8 4 52.4	775.920408	1872.0	Eichelberger
K.....	1875 Jan. 0.0....Greenwich........	8 4 54	775.9214	1875.0	Leuschner

(94) AURORA

Discovered by Watson[1], 1867, September 6, at Ann Arbor.

Preliminary elements were computed by Tietjen[2,3], the second set given below as Elements A.

With Elements A, H. Leppig[4] formed eight normal places over 162 days, and determined Elements B. An accurate ephemeris including special perturbations by Encke's method was computed for 1868, December 15 to 1869, January 31. Observations by Vogel, January 15, 17, 18, 1869, gave residuals, $\Delta\alpha$ $-21^s.6$, $\Delta\delta$ $+53''.5$.

Thereafter, the B. J. gives elements by Leppig from 1872 to 1915, (but with a misprint of 4° in ω from 1887 to 1897[5]). Bauschinger gives Leppig's Elements C in the "Tabellen"[6] for the epoch 1883, July 12.0. The various elements in the B. J. to 1915 are probably brought up from Leppig's Elements B, with special perturbations. The character of the perturbations is not given. Nor is any reference made to arbitrary corrections.

In Kleine Planeten, 1916, the elements are changed by estimating the perturbations 1883–1910 and roughly determining, M and μ, Elements D.

In Kleine Planeten, 1921, the last elements are again corrected by the computation of Jupiter's perturbations and a representation of observed positions, 1884-1918, within $\pm 1^m.5$. Osculating elements, 1921, April 24, not available.

From 1884 to 1899 the planet was practically lost, mainly on account of the misprint in ω. Coddington[7] computed a place with the elements of B. J. 1901, and found the planet $\Delta\alpha$ $+5^m.2$, $\Delta\delta$ $-20'$, 1899. For the computation of the perturbations and tables of the Watson asteroids, Leuschner made a collection of Leppig's elements in the B. J. including Elements C and derived average Elements E from those in B. J. 1871, 1873, 1874, 1875, 1877, 1878, 1881, 1898. From these, approximate mean elements were derived. The perturbations were developed and a preliminary correction of the mean motion and mean anomaly from observations in 1867 and 1899 was attempted. A mistake in one of the main terms of the perturbations was discovered and corrected. Nevertheless, large discrepancies between observation and computation remained ($-5°$ in α, 1867). These differences were used for a preliminary correction of the mean motion and of the mean anomaly. With the new value of the mean motion the perturbations were corrected, the residuals re-determined, and further corrections made to the mean motion and to the mean anomaly. A least square solution of 12 places from 1867 to 1899 was then made including the corrected perturbations. The residuals were thereby reduced from ±2°.2 to a maximum of ±0°.14. Elements F.

Further correction of the perturbations by means of the new mean motion produced larger residuals. The best representation as above is obtained by the use of the adopted Elements F without further correcting the perturbations.

From the experience of the Recheninstitut and of Leuschner with Leppig's elements, it is evident that before a satisfactory representation for all oppositions, without making arbitrary corrections to the elements, can be obtained, it will be necessary to derive an accurate set of osculating elements by connecting a limited number of oppositions with accurate determination of the perturbations. With an accurate set of osculating elements the perturbations may then be corrected, but further correction of the elements should be made only after higher order perturbations and perturbations by planets other than Jupiter shall have been considered.

REFERENCES

[1] A. N. vol. 70, p. 79.
[2] A. N. vol. 70, p. 219.
[3] A. N. vol. 71, p. 47.
[4] B. J. 1871. A. N. vol. 72, p. 331.
[5] A. N. vol. 139, p. 63.
[6] Tabellen zur Geschichte und Statistik der Kleinen Planeten.
[7] A. N. vol. 153, p. 225.

TABLE 8.—*Elements—(94) Aurora*

Letter	Epoch	M.T.	M	ω	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A.....	1867 Nov. 28.0..	Berlin......	340 30 39.5	40 50 23.2	4 32 9.3	8 5 27.0
B.....	1870 Jan. 0.0....	Berlin......	115 9 43.74	40 2 43.08	4 34 36.38	8 5 18.49
C.....	1883 July 12.0..	Berlin......	256 3 4.3	45 22 31.8	4 25 0.9	8 4 14.0
D.....	1925 Jan. 0.5....	Greenwich..	20 40 1.2	57 20 9.6	4 22 26.4	8 4 8.4
E.....	1875 Jan. 0.0...	Greenwich..	73 37 45	41 51 21	4 28 46	8 4 54
F.....	1875 Jan. 0.0...	Greenwich..	65 20 12	48 13 54	4 24 21	8 3 51

Letter	Epoch	M.T.	φ	μ	Equinox	Author
			° ′ ″	″		
A......	1867 Nov. 28.0	Berlin......	5 10 18.6	630.5129	1867.0	Tietjen
B......	1870 Jan. 0.0........	Berlin......	5 6 8.1	631.5264	1870.0	Leppig
C......	1883 July 12.0........	Berlin......	4 44 18.3	630.6584	1900.0	Leppig
D......	1925 Jan. 0.5........	Greenwich..	5 4 58.8	631.800	1925.0	Berberich
E......	1875 Jan. 0.0........	Greenwich..	4 56 22	631.2196	1900.0	Leuschner
F......	1875 Jan. 0.0........	Greenwich..	5 17 16	631.9473	1900.0	Leuschner

(127) JOHANNA

Discovered by Prosper Henry[1] at Paris, 1872, November 5.

In B. J. 1875 a set of Elements A by Baillaud is published, based on observations Nov. 9, 22, 28.

Preliminary Elements B by Renan[2] are based on seven observations for the opposition 1872–73. An ephemeris for 1874 April and May is also published. An observation on April 17, 1874, shows corrections to the ephemeris $\Delta\alpha$ $+2^m43^s$, $\Delta\delta$ $-16'$.

An improvement on Elements B is made by Renan[3] on the basis of six normal places (1872–1874). The resulting Elements C represent the normal places within $+7''.8$ and $-9''.1$. Renan states these differences are within the limits of error and Elements C may be considered definitive. An ephemeris is then computed for 1876 September.

Elements D are published by Bauschinger.[4] They are by Maywald and are based on oppositions 1872, 1874, 1876, 1879. The special perturbations of Jupiter and Saturn to 1890 are included.

Renan's elements are published and used by B. J. 1882 to 1887. Beginning with B. J. 1888, Maywald's elements are used. An empirical correction was applied to the mean anomalie in 1913 by Berberich.[5]

A correction to the R. I. ephemeris[6] for 1918, November 23, is $\Delta\alpha$ $-5^m.2$, $\Delta\delta$ $-20'$. An improved ephemeris is published[7] for the period 1920, March 21 to April 21, from elements based on 6 oppositions, 1897–1908. Jupiter's perturbations are included. Representation $\pm 0^m.3$. Osculation 1921, July 13. The correction to this ephemeris[6] from an observation[8] February 28, 1920, is $\Delta\alpha$ $+0^m.4$, $\Delta\delta$ $-6'$.

Another ephemeris[9] for 1920 by P. Maitre is based on elements published in Connaissance des Temps for 1915 with the mean anomaly corrected, on the basis of observations in 1917 and 1918, ΔM $-1^\circ.305$.

General perturbations applying Bohlin's method for this planet have been published by D. T. Wilson[10] and similar perturbations with Hansen's method, by M. Viljev.[11]

Olson[12] has published general perturbations of the first order by Jupiter. The basic elements are those of Maywald. Jupiter mass 1:1047.568 (Bessel-Schur). Method that of Hansen. Terms of 6th to 8th order are below 1″. No comparison with observations.

REFERENCES

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[2] A. N. vol. 83, p. 349; C. R. vol. 78, 1874, p. 1219.
[3] C. R. vol. 83, 1876, p. 567.
[4] Veröffentlichungen R. I. No. 16.
[5] B. J. 1915, p. 27.
[6] A. N. vol. 208, p. 14.
[7] B—Z der A. N. No. 11, 1920.
[8] B—Z der A. N. No. 14, 1920.
[9] C. O. M. No. 297.
[10] Ast. Iakttagelser och Undersökningar, Stockholm. vol. 10, No. 1.
[11] Bull. Soc. Astr. Russio. No. 22.
[12] Swed. Akad. Handl., Stockholm, 1895.

TABLE 9.—*Elements—(127) Johanna*

Letter	Epoch	M.T.	M	ω	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A......	1872 Dec. 18.0...........	Berlin......	293 6 46	90 25 17	31 40 11	8 19 42
B......	1874 Apr. 17.0...........	Berlin......	35 3 42	91 13 48	31 41 41	8 17 28
C......	1876 Sept. 5.5...........	Berlin......	223 47 46	90 50 37	31 46 38	8 16 40
D......	1879 Apr. 4.0...........	Berlin......	67 49 52	89 18 47	31 45 2	8 16 48

Letter	Epoch	M.T.	φ	μ	Equinox	Authority
			° ′ ″	″		
A......	1872 Dec. 18.0............	Berlin......	4 36 31	766.23	1872.0	Baillaud
B......	1874 Apr. 17.0............	Berlin......	3 35 48	776.368	1874.0	Renan
C......	1876 Sept. 5.5............	Berlin......	3 46 51	775.9173	1880.0	Renan
D......	1879 Apr. 4.0............	Berlin......	3 51 17	775.7686	1880.0	Maywald

(128) NEMESIS

Discovered by Watson[1] at Ann Arbor on November 25, 1872, and also by Borrelly[2] at Marseilles on December 4, 1872.

Preliminary Elements A were computed by Bossert,[3] based on observations 1872, November 25, December 7, and December 22. He also published a short search ephemeris.

Preliminary Elements B based on observations covering the first five months were published[4] by Leo de Ball. He then forms six normal places from the same series of observations and from these determines Elements C. These elements represent the normal places within $-2''.8$ and $+3''.2$. An ephemeris for 1874 is given in the same reference, page 374.

Preliminary Elements D, based on observations 1872, November 25, December 4, and December 12, are published by H. Richter.[5]

Elements E, based on eight normal places from the first two oppositions (1872–1874) were computed by Ball.[6] The normal places are represented in the plane between $+2''.13$ and $-2''.54$ and perpendicular to the plane between $+2''.7$ and $-2''.3$. The special perturbations of Jupiter and Saturn were taken into consideration.

Further Elements F for a new epoch and mean equinox, in which the special perturbations due to Jupiter and Saturn have been taken into account, have been published by Ball[6] and also an ephemeris for 1875.

The maximum correction to the ephemeris[7] computed from Ball's elements, for July 1880, is $\Delta\alpha$ $+2^s.14$ and $\Delta\delta$ $+12''.0$.

Elements G by Palisa are published and used in B. J. 1883, to B. J. 1893. They are based on 5 oppositions 1872–79 and include the perturbations by Jupiter and Saturn.

Ball's elements are again published and used in B. J. 1894 (see Elements H) to B. J. 1913.

Empirical corrections[8] were applied to Ball's Elements H,[9] by Berberich.

The most extensive investigation on this planet is by Leuschner.[10] The Elements I are based on observations extending from 1872 to 1899 and include the general perturbations due to Jupiter of the first order. These elements and perturbations are used in the B. J. 1915 and to date.

The correction to the ephemeris[11] based on Elements I on June 16, 1921, was $\Delta\alpha$ $+0^m.6$ and $\Delta\delta$ $-1'$. Corrections to these elements should be applied only on the basis of higher order perturbations by Jupiter and of perturbations by other planets.

REFERENCES

[1] A. N. vol. 80, p. 319.
[2] A. N. vol. 80, p. 297.
[3] A. N. vol. 80, p. 351.
[4] No. 5, Circular der Berliner Sternwarte. A. N. vol. 82, p. 281.
[5] A. N. vol. 82, p. 63.
[6] A. N. vol. 85, p. 331; vol. 86, pp. 29–31.
[7] A. N. vol. 98, p. 55, and vol. 99, p. 251.
[8] A. N. vol. 189, p. 173.
[9] B. J. 1914.
[10] Mem. of the National Academy of Sciences. vol. x, seventh mem., p. 255.
[11] C. O. M. No. 181.

TABLE 10.—*Elements—(128) Nemesis*

Letter	Epoch	M. T.	M	ω	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A.....	1873 Jan. 1.0	Greenwich	49 56 20	296 1 28	76 35 50	6 18 27
B.....	1873 Feb. 25.5	Berlin....	59 55 44	298 35 26	76 39 55	6 15 9
C.....	1873 Feb. 25.5	Berlin....	59 57 21	298 34 00	76 40 02	6 15 14
D.....	1872 Nov. 25.0	Berlin....	62 39 10	274 9 56	75 12 46	7 30 40
E.....	1873 Feb. 25.5	Berlin....	59 52 17	298 41 4	76 37 42	6 15 14
F.....	1875 Apr. 25.0	Berlin....	228 43 54	300 3 32	76 30 40	6 15 31
G.....	1880 July 7.0	Berlin....	279 16 11	300 16 4	76 32 4	6 15 43
H.....	1892 Feb. 15.0	Berlin....	116 23 35	299 30 58	76 36 54	6 15 24
I......	1896 July 3.0	Berlin....	101 41 9*	299 56 32	76 39 30	6 15 18

Letter	Epoch	M. T.	φ	μ	Equinox	Authority
			° ′ ″	″		
A.....	1873 Jan. 1.0...	Greenwich	7 21 59	776.86	1873.0	Bossert
B.....	1873 Feb. 25.5...	Berlin....	7 13 5	778.030	1873.0	De Ball
C.....	1873 Feb. 25.5...	Berlin....	7 12 51	778.1516	1873.0	De Ball
D.....	1872 Nov. 25.0...	Berlin....	7 13 17	764.990	1872.0	Richter
E.....	1873 Feb. 25.5...	Berlin....	7 11 59	778.0333	1870.0	De Ball
F.....	1875 Apr. 25.0...	Berlin....	7 13 20	777.4729	1875.0	De Ball
G.....	1880 July 7.0...	Berlin....	7 21 52	777.4964	1880.0	Palisa
H.....	1892 Feb. 15.0...	Berlin....	7 10 53	777.6921	1890.0	De Ball
I......	1896 July 3.0...	Berlin....	7 16 50	777.8761*	1900.0	Leuschner

*Mean Elements.

(175) ANDROMACHE.

Discovered by J. Watson[1] 1877 October 1, but observed only October 5, 6, 16, 29. These were all the observations available until rediscovery May 19, 1893.

By an unfortunate delay the news did not reach other interested observatories until two months later. The preliminary orbit, Elements A, by Tietjen,[2] was erroneous, partly on account of errors in Watson's ringmicrometer observations.

Watson[3] computed Elements B, and with them perturbations for three years until his death in 1880.

Bidschof[4] made an attempt to improve the orbit, Elements C, but perceived the impossibility of this undertaking.

1893, May 19, Charlios[5] at Nice discovered by photography a planet, 1893 Z, and noticed the close similarity between its orbit and that of (175) Andromache. He referred the case to Berberich[6] who followed it out from a preliminary orbit to the best to be obtained from the data in 1893 May 19-August 1, Elements D, and compared his results with the observations in 1877 and with one in 1892 from a photographic plate taken at Heidelberg.

Berberich[7] next computed the extremely large perturbations by Jupiter (in 1887) and Saturn, and improved the orbit by a solution from four normal places in 1877, 1892, 1893, with an ephemeris for the coming opposition, Elements E—"this planet therefore deserves peculiar attention for it will furnish an excellent means for determining an accurate value of the mass of the planet Jupiter." Cf. (3) Juno, (4) Vesta, (13) Egeria, (24) Themis, (33) Polyhymnia, (447) Valentine.

Berberich[8] improved his elements by the following oppositions and brought them forward with the special perturbations of Jupiter and Saturn. The Elements F, G, H, illustrate the large changes in this case of near commensurability (Hecuba group).

This case among others impelled A. O. Leuschner to undertake the computation of the tables[9] for the Hecuba group after Bohlin's method. The application of these tables to the orbit of (175) Andromache was carried out by Miss S. H. Levy. Her unpublished computation contains the transformation of Berberich's elements to Mean Elements I, tables of the perturbations, determination of constants, the comparison with ten oppositions between 1893 and 1907, and at least squares solution which led to the Mean Elements J. The representation of observations in 1914 was $-0^m.4$ in α and $+1'$ in δ.

The comparison between theory and observation has been discussed by A. O. Leuschner.[10]

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[2] Circular zum B. J. Nr. 81.
[3] B. J. 1881, 1882, 1886.
[4] Sitz-Ber. Akad. Wien. Bd. 98.
[5] A. N. vol. 132, p. 367.
[6] A. N. vol. 134, p. 143.
[7] A. J. vol. 14, p. 36.
[8] A. N. vol. 153, p. 59.
[9] National Academy of Sciences, Memoirs, vol. 14, third memoir.
[10] A. O. Leuschner, Comparison of theory with observation for the minor planets (10) Hygiea and (175) Andromache with respect to the perturbations by Jupiter. Proc. N. A. S. Washington. vol. 8, No. 7, p. 170.

Table 11.—*Elements—(175) Andromache*

Letter	Date	M. T.	M	ω	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A......	1877 Oct. 29.5....	Berlin.............	45 6 25.6	269 26 21.1	23 32 56.0	3 46 38.8
B......	1878 Dec. 5.0....	Berlin.............	105 28 46.7	269 31 45.6	23 34 50.2	3 46 36.7
C......	1889 Apr. 7.5....	Berlin.............	317 3 18.5	269 42 7.7	23 43 24.9	3 46 45.8
D......	1893 Aug. 1.5....	Berlin.............	297 57 33.9	299 49 1.8	25 27 14.8	3 10 51.4
E......	1894 Aug. 23.0....	Berlin.............	3 52 54.5	299 46 4.9	25 27 32.5	3 10 59.4
F......	1877 Oct. 11.0....	Berlin.............	32 5 0.1	298 31 16.6	25 36 3.8	3 11 42.8
G......	1900 Sept. 1.0...	Berlin.............	16 10 41.5	301 33 8.5	25 23 37.7	3 10 38.9
H......	1920 Jan. 20.....	1925.0 Greenwich..	76 24 14	306 45 58	25 7 12	3 10 37
I.......	1877 Oct. 11.0....	Berlin.............	3 49 12	304 6 54	25 36 4	3 11 43
J.......	1877 Oct. 11.0....	Berlin.............	3 59 32	304 13 9	25 43 27	3 11 29

Letter	Date	M. T.	φ	μ	Equinox	Authority
			° ′ ″	″		
A......	1877 Oct. 29.5....	Berlin.............	20 26 45.7	542.173	1877.0	Tietjen
B......	1878 Dec. 5.0....	Berlin.............	20 26 12.6	541.779908	1880.0	Watson
C......	1889 Apr. 7.5....	Berlin.............	20 15 17.8	544.411	1890.0	Bidschof.
D......	1893 Aug. 1.5....	Berlin.............	11 39 45.8	614.943	1890.0	Berberich
E......	1894 Aug. 23.0....	Berlin.............	11 36 51.4	614.63354	1890.0	Berberich
F......	1877 Oct. 11.0....	Berlin.............	12 8 54.1	617.7375	1890.0	Berberich
G......	1900 Sept. 1.0...	Berlin.............	11 7 42.9	612.2868	1900.0	Berberich
H......	1920 Jan. 20.....	1925.0 Greenwich..	10 42 11	607.899	1925.0	Berberich
I.......	1877 Oct. 11.0....	Berlin.............	12 21 26	619.5629	1890.0	Berberich
J.......	1877 Oct. 11.0....	Berlin.............	12 19 34	619.025	1890.0	Miss Levy

(433) EROS, 1898 DQ.

Discovered 1898, August 13, by Witt at Berlin and by Charlois at Nice.

Fayet[1] computed Elements A from three observations on August 15, 26, and September 7. Later he[2] computed Elements B from normal places 1898 August 16.5(7 observations), September 17.5(6 observations), and October 22.4(2 observations). The residuals vary from $-0^s.11$ to $-0^s.36$ in α, and $+5''.1$ to $+8''.2$ in δ.

Hussey[3] gives Elements C, computed from the mean of two Kiel observations made on the 15th of August, and observations made at Mt. Hamilton on September 6 and 27. Hussey[4] later computed Elements D from observations at Mt. Hamilton on August 15, September 27, and November 11, 1898. The residuals for the observations in 1898 vary from $+0^s.04$ to $-0^s.18$ in α, and $+2''.2$ to $+4''.4$ in δ and observations to May 4, 1899, are closely represented.

Chandler[5] computed Elements E from observations August 14 to November 16, 1898. Elements F[6] and G[7] are also due to Chandler. Elements F are from eight normal places from August 17.5 to November 26.5. The representation is within the errors of observation.

In a later article he gives the following residuals for earlier observations found on plates taken in 1896 at Arequipa:

1896	$\Delta\alpha$	$\Delta\delta$
April 6	$-0^m\ 36^s.0$	$-4'.9$
June 5	$-1\ \ 16.7$	$-5'.8$

Elements G[7] are the preceding ones with corrections applied so as to fit the observations made at Arequipa 1896. Representation[8] for 1893, 1894, 1896, gives maximum residual of $+7^s.6$ in α and $-1'.0$ in δ.

The observations from 1893 to February 16, 1894, were found by Pickering and Mrs. Fleming on plates taken at Cambridge. Perturbations were not considered in applying the corrections to Elements F to obtain Elements G.

Elements H are due to Berberich[9] and are given under the title of "First elements," and are based upon observations made at Urania (Berlin) on the 14, 23, and 31 of August. Berberich gives forty-three sets of residuals covering the period August 13 to August 31, $\Delta\alpha$ being greater than $0^s.40$ but three times, and $\Delta\delta$ being larger than $10''.0$ but twice.

Elements I[10] J[11] K[12] and L[13] are due to Russell. Elements I are computed from three normal places obtained by comparison of observations in the A. N. and A. J. with places computed from Berberich's Elements H.[9] (1898, August 18.5, 34 observations, August 26.5, 16 observations, and September 9.5, 26 observations, heliocentric arc about 8°.) The set J[11] is based upon nine normal places, although Dr. Chandler's value of the mean motion was taken (Elements G) and the other elements determined by varying the ratio of the extreme geocentric distances. The normal places are well represented.

Elements K, Russell,[12] are the same as J[11] except for changes in M, ω, and ϕ based upon observations in 1899.0 at the Chamberlin and Lick Observatories. The representation August 17, 1898, to May 20, 1899, is satisfactory except for the normal place of November 11.5 for which $\Delta\alpha$ —^{s}.28, $\Delta\delta$ —3″.

Elements L are merely the preceding ones brought up to the epoch 1900.0 and mean equinox of 1900.0. In his article [13] Russell develops the general perturbations of the major axis of Eros by the action of Mars. He does this by Le Verrier's method of interpolation. Russell finds eight terms of the general perturbations of the mean longitude larger than 1″.50. The largest is 35″ with a period of about 1000 years. The greatest displacement due to the first 7 terms will be +38″ in 1927 and —53″ in 1959 in mean longitude, and "will eventually lead to a valuable determination of the mass of Mars." Russell then gives tables of the perturbative function, perturbations of log a and perturbations of the mean longitude.

Elements M [14] were developed by Robbins from Elements G by applying special perturbations of Venus, Earth, Mars, Jupiter, and Saturn by the method of the variation of constants. (Nautical Almanac 1837, appendix.)

Elements N[15] are due to H. Osten, who has computed eight normal places based upon Elements P, with special perturbations of Venus, Earth, Mars, Jupiter, and Saturn according to Encke's method.

Millosevich has produced numerous sets of elements. Elements O[16] were computed from observations made during the interval August 14 to September 21, 1898. An observation by Millosevich October 8, gives $\Delta\alpha$ —1^s.93, $\Delta\delta$ +7″.5.

Elements P[17] were computed from a normal place of date August 14.5 and Millosevich's observations on September 21 and October 24, 1898. Millosevich states that Berberich's ephemeris requires a correction of +131^s in α and +5′.5 in δ on December 23, 1898.

Elements Q[18] are from photographic observations at Greenwich by the variation of the distances. Later [19] he stated that there is an error of about 2^s in his ephemeris after five months.

Elements R[19] are based upon four normal places and are Elements P improved by the method of variation of the distances. They represent 17 normal places from 1000 observations in 1898–1899 perfectly.

Elements S[20] are based upon 17 normal places made from 999 observations in α and 992 observations in δ in the years 1898–1899, and are derived from Elements R brought forward with the perturbations of Venus, Earth, Mars, Saturn and Jupiter for 20-day intervals.

Elements T[21] U[22] V[23] W[24] X[25] Y[26] are improvements of preceding elements, as are Elements Z[27] AA[27] AB[27] AC[28] AD[29] and AE[30]. Elements AE are Elements AC with special perturbations for 20-day intervals for the period 1901, March 20, to 1903, June 8.0 applied. These perturbations were computed by Wedemeyer, those of Venus, Earth, Mars, Jupiter and Saturn being considered.

Elements AF were found only in the B. J. for 1907. They are probably Elements AE with the epoch changed and perturbations applied. Dziewulski has published "Sekulare Marsstörungen in der Bewegung des Eros.," Bull. de l'Acad. des Science de Cracovie 1905. Not available here. Some mistakes are corrected in A. N. vol. 175, p. 171. In A. N. vol. 175, p. 17, Merfield gives the secular perturbation of Eros due to all major planets. Gauss' method by Hill. Elements from Hill's memoir, Ast. Papers of the Amer. Ephm. Vol. IV and Elements W. The secular perturbations by Jupiter, the Earth and Venus are the largest.

Elements AG are by G. Witt.[31] They are based upon observations from 1893 to 1903, the perturbations of Venus, Earth, Mars, Jupiter and Saturn being included. The perturbations were calculated by the method of variation of constants. For Mars, Jupiter and Saturn the perturbations were calculated for 20-day intervals and for Venus and the Earth at 10-day intervals. With these elements he calculates the perturbations from 1903 to the beginning of 1908, and includes these in an ephemeris for 1905 for dates from July 17 to August 22 (B. J. 1907, p. 476.).

Elements AH[32] AI[33] AJ[34] are preceding elements with change of osculation. Elements AI are also found in the Connaissance des Temps for 1915, but with the equinox changed in 1920.0, Greenwich M. T.

In an article "Beiträge zur Theorie der Bewegung des Planeten 433 Eros," E. Noteboom[35] uses Elements AL to compute the general perturbations of Mercury, Uranus, and Nepture. A recurring run in the residuals is not due to these planets.

Noteboom then uses the twenty normal places of Witt[36] (which lie between dates 1893 October 31 and 1907 October 8) and forms four more normal places, one in 1910, one in 1912, and two in 1914. He then gets Elements AM out of a least square solution by a correction of Elements AL. He gives the mass of Earth and Moon as $1/328370 \pm 102$ and $\pi = 8''.799$.

No authority is available for Elements AK.[37]

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[32] B. J. 1908, p. 496
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[36] "Über die Notwendigkeit einer Verbesserung der Masse des Systems Erde-Mond," by G. Witt. Vierteljahrsschrift der A. G. 1908, p. 295.
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TABLE 12.—*Elements—(433) Eros*

Letter	Epoch	Mean Time	M	ω	Ω	i	φ	μ	log a	Authority	Equinox
			° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
A.....	1898 Aug. 15.5..	Paris.......	221 25 0	172 7 7	302 25 38	9 57 35	12 12 14	2018.45	0.163326	G. Fayet	1898.0
B.....	1898 Aug. 16.5..	Paris.......	213 1 42.4	177 46 7.5	303 30 19.8	10 49 36.7	12 51 53.9	2016.11	0.163662	G. Fayet	1898.0
C.....	1898 Aug. 31.5..	Greenwich..	222 51 53.3	176 52 17.6	303 23 45.2	10 44 41.3	12 49 40.7	2013.491	0.164038	W. Hussey	1898.0
D....	1898 Aug. 31.5..	Greenwich..	221 33 29.10	177 41 21.00	303 30 19.2	10 49 33.5	12 52 16.1	2015.775	0.1637097	W. Hussey	1898.0
E.....	1898 Aug. 31.5..	Greenwich..	219 59 23.8	178 40 7.3	303 36 1.1	10 50 31.2	12 51 1.6	2023.656	0.162580	S.C.Chandler	1898.0
F.....	1898 Aug. 31.5..	Greenwich..	221 45 45.6	173 33 32.5	303 29 45.4	10 49 27.2	12 52 32.8	2014.6326	0.1638739	S.C.Chandler	1898.0
G....	1898 Aug. 31.5..	Greenwich..	221 35 45.6	177 37 56.0	303 31 57.1	10 50 11.8	12 52 9.8	2015.2326	0.1637876	S.C.Chandler	1898.0
H....	1898 Aug. 31.5..	Berlin......	220 14 3.7	178 28 26.2	303 48 53.0	11 6 57.1	13 13 3.8	2010.131	0.164521	Berberich	1898.0
I.....	1898 Aug. 31.5..	Berlin......	224 33 12.3	175 47 50.1	303 20 20.3	10 45 1.8	12 55 13.6	2003.86	0.1654245	Russell	1898.0
J.....	1898 Aug. 31.5..	Greenwich..	221 38 37.8	177 38 15.2	303 29 57.3	10 49 31.0	12 52 27.9	2015.2326	0.1637876	Russell	1898.0
K....	1898 Aug. 31.5..	Greenwich..	221 37 2.0	177 39 10.6	303 29 57.3	10 49 31.0	12 52 14.2	2015.2326	0.1637876	Russell	1898.0
L.....	1900 Jan. 0.....	Greenwich..	133 57 12.7	177 39 6.8	303 31 41.5	10 49 31.6	12 52 14.2	2015.2326	0.1637876	Russell	1900.0
M....	1900 Nov. 1.5..	Greenwich..	304 57 42.8	177 38 2.9	303 33 18.5	10 50 16.5	12 52 38.3	2015.2423	0.1637863	F. Robbins	1900.0
N....	1898 Oct. 1.0..	Berlin......	238 39 44.64	177 39 21.05	303 31 53.37	10 49 33.99	12 52 18.33	2015.34326	....	H. Osten	1900.0
O.....	1898 Aug. 31.5..	Berlin......	222 23 28.7	177 9 34.8	303 24 53.1	10 45 18.1	12 49 5.4	2015.119	0.163804	Millosevich	1898.0
P.....	1898 Aug. 31.5..	Berlin......	222 16 27.1	177 13 32.6	303 27 48.3	10 48 33.5	12 52 48.3	2012.2186	0.1642210	Millosevich	1898.0
Q.....	1898 Aug. 31.5..	Berlin......	221 40 29.6	177 36 7.3	303 29 50.9	10 49 33.1	12 52 25.4	2014.656	....	Millosevich	1898.0
R....	1898 Aug. 2.5..	Berlin......	205 22 27.98	177 38 39.02	303 31 46.59	10 49 35.30	12 52 21.99	2015.16324	0.1637975	Millosevich	1900.0
S.....	1900 Oct 31.5..	Berlin......	304 23 59.7	177 38 41.6	303 30 40.4	10 49 38.9	12 52 48.2	2015.12740	0.1638027	Millosevich	1900.0
T.....	1898 Aug. 2.5..	Berlin......	205 21 13.79	177 39 18.84	303 31 44.14	10 49 29.36	12 52 5.98	2015.33037	....	Millosevich	1900.0
U....	1900 Oct. 31.5..	Berlin......	304 25 2.56	177 39 21.41	303 30 37.99	10 49 32.98	12 52 32.15	2015.29453	....	Millosevich	1900.0
V.....	1898 Aug. 2.5..	Berlin......	205 21 41.83	177 38 55.23	303 31 56.17	10 49 35.35	12 52 14.44	2015.26908	0.1637824	Millosevich	1900.0
W....	1900 Oct. 31.5..	Berlin......	304 24 40.34	177 38 57.80	303 30 50.02	10 49 38.97	12 52 40.61	2015.23324	0.1637875	Millosevich	1900.0
X....	1898 Aug. 2.5..	Berlin......	205 21 42.96	177 39 3.61	303 31 51.72	10 49 35.36	12 52 21.16	2015.27302	0.1637818	Millosevich	1900.0
Y....	1900 Oct. 31.5..	Berlin......	304 24 44.71	177 39 6.18	303 30 45.57	10 49 38.98	12 52 47.33	2015.23718	....	Millosevich	1900.0
Z.....	1900 Oct. 31.5..	Berlin......	304 24 49.75	177 39 8.65	303 30 42.38	10 49 38.03	12 52 49.93	2015.23858	0.1637867	Millosevich	1900.0
AA...	1900 Dec. 10.5..	Berlin......	326 48 27.27	177 39 13.04	303 30 33.08	10 49 38.75	12 52 53.47	2015.18174	0.1637950	Millosevich	1900.0
AB...	1901 Feb. 8.5..	Berlin......	0 23 41.36	177 39 12.65	303 30 28.04	10 49 39.89	12 53 0.33	2015.05612	0.1638129	Millosevich	1900.0
AC...	1901 Mar. 20.5..	Berlin......	22 47 1.51	177 39 17.51	303 30 27.30	10 49 39.54	12 53 1.03	2015.04007	0.1638153	Millosevich	1900.0
AD...	1901 Mar. 20.5..	Berlin......	22 46 55.3	177 39 12.4	303 30 31.6	10 49 40.4	12 53 0.5	2014.98718	0.1638229	Millosevich	1900.0
AE...	1903 June 8.0..	Berlin......	115 47 54.3	177 43 17.2	303 30 4.2	10 49 41.9	12 53 21.9	2015.2118	0.1637906	Millosevich	1900.0
AF...	1905 Aug. 6.0..	Berlin......	197 55 55.8	177 24 21.9	303 37 35.5	10 49 39.5	12 52 53.8	2014.9884	0.1638227	Millosevich	1910.0
AG...	1898 Aug. 2.0..	Berlin......	205 4 51.833	177 39 11.209	303 31 48.211	10 49 35.040	12 52 24.153	2015.275469	....	G. Witt	1900.0
AH...	1905 Aug. 6.0..	Berlin......	197 55 57.2	177 44 21.4	303 37 36.8	10 49 39.0	12 52 54.1	2014.9849	0.1638232	G. Witt	1910.0
AI....	1907 Oct. 15.0..	Berlin......	285 40 28.0	177 46 3.8	303 37 3.5	10 49 41.2	12 52 58.8	2015.0581	0.1638127	G. Witt	1910.0
AJ....	1914 Sept. 28.0..	Berlin......	267 11 1.2	177 50 23.3	303 35 8.6	10 49 39.6	12 53 0.5	2014.8293	0.1638457	G. Witt	1910.0
AK...	1925 Jan. 0.5..	Greenwich..	204 35 6.0	177 49 55.2	303 48 10.8	10 49 44.4	12 52 58.8	2014.829	....	....	1925.0
AL...	1898 Aug. 2.0..	Berlin......	205 4 51.547	177 39 11.579	303 31 47.859	10 49 35.237	12 52 24.388	2015.275271	....	G. Witt	1900.0
AM...	1898 Aug. 2.0..	Berlin......	205 4 51.429	177 39 11.419	303 31 48.405	10 49 35.138	12 52 24.579	2015.274706	....	E. Noteboom	1900.0

(447) VALENTINE (1899 ES).

Discovered by Wolf and Schwassmann[1] at Heidelberg 1899, October 27.

Preliminary elements were computed by Kreutz[2] based on observations 1899, October 29, November 11 and December 3. Elements A.

Corrections to the ephemerides based on improved elements in 1902[3] $\Delta\alpha$ $+16^s$, $\Delta\delta$ $-2'$; in 1905[4] January 13, $\Delta\alpha$ $+36^s$, $\Delta\delta$ $-0'.3$.

A more complete investigation of the orbit was undertaken by Hans Osten.[5] He received from Kreutz five sets of elements, B to F, which refer to different epochs in order to show the effects of the perturbations. Kreutz states that Elements B to E are comparable and are to be preferred to Elements F. Elements B to E were determined from five normal places (1899 to 1904). The residuals for the normal places are

	Ia	Ib	II	III	IV	V
$\Delta\alpha \cos\delta$	+5.″1	+8.″7	+1.″2	−3.″2	−0.″7	+35.″5
$\Delta\delta$	+1.4	+8.1	−1.1	+4.0	−0.2	− 3.0

For the purpose of investigating the general perturbations for this planet, Osten makes use of Kreutz' Elements B, Leverrier's elements for Jupiter and Saturn, except for the adoption of Hill's values of the mean motion for each, for Jupiter's mass Newcomb's value and for Saturn's mass Bessel's value. The perturbations of the first order of the masses are determined according to Hansen's method. As a test on his results, he compares observations with theory for seven normal places (1894 to 1906) with the following results:

	$\Delta\alpha \cos\delta$	$\Delta\delta$
1. 1894	−10.″73*	− 0.″54*
2. 1899a	+ 5.07	+ 1.41
3. 1899b	+ 8.28	+ 8.06
4. 1901	+ 0.32	— 1.01
5. 1902	+ 2.44	+ 1.39
6. 1904	+33.26	+16.12
7. 1906	+32.06	— 2.52

On the basis of these residuals the elements were corrected, which resulted in Elements G. The comparison between observation and computation for the normal places gives:

* Weight 1/9. Observation uncertain.

	$\Delta\alpha \cos\delta$	$\Delta\delta$	
1. 1894	+8.76†	+7.22†	
2. 1899a	—2.81	—2.36	‡
3. 1899b	+1.57	+4.86	‡
4. 1901	—2.51	+0.73	‡
5. 1902	+1.86	+2.00	‡
6. 1904	—0.68	—0.31	‡
7. 1906	+1.97	—1.05	‡

The approximate perturbations due to Mars were found to be ineffective since they remained less than 0″.01.

For the purpose of improving Elements G by means of additional observations, Osten[6] first computes special perturbations due to Jupiter and Saturn by the method of variation of elements, and determines new Mean Elements H from the corrections which the observations successively indicate. For the other six major planets the general perturbations are computed after the method in Tisserand, Vol. 1, Chap. 22. All the masses are taken from Bauschinger, "Tafeln z. Theor. Astr." From this combination of special and general perturbations a set of osculating elements for the ten oppositions 1899–1912 results, forming the basis for the formation of normal places. A detailed investigation of the comparison stars and the observations leads to normal places with relative weights and corrections for magnitude equation in α. The equations of conditions are solved under four different assumptions, thus Elements I without and Elements J with magnitude equation. No definite solution is accepted, but the importance of using α observations free from magnitude equation is emphasized.

In the next work Osten[7] proceeds to determine the perturbations of the second order starting with Elements I. Hansen's method is followed throughout (correcting an error in "Auseinandersetzung II, page 98," which acts nearly as a change of the perturbing mass). A capital difficulty is encountered in the near commensurabilities ($+2\epsilon - 5\epsilon' + 1\epsilon''$) with a period of 8650 years.

Osten proposes to treat such inequalities by expansion into power series in the time and then eliminate one of the anomalies. For (447) the term $3\epsilon - 7\epsilon'$ is thereby much enlarged. A comparison is made between the special and general perturbations by Jupiter and Saturn. Some deviations of the order of 1″ are attributed to the perturbations of the third order. Eight tables contain the perturbations.

In the hope of obtaining as accurate results for (447) as for the major planets Osten[8] completes his former theory and gives the last

† Uncertain.
‡ Normal places from Kreutz.

part of the perturbations of the second order and those of the third in longitude and radius vector. The accuracy of 1″ in 100 years is extremely difficult to obtain. A first test is the comparison between special and general perturbations in the longitude and radius vector. He finds small deviations which probably are due to the computation of the special perturbations.

As a further test Osten gives the comparison with 16 normal places 1894–1918. The Jupiter mass is also included as an unknown. The value 1:1047.49 is found. 1:1047.35 according to Newcomb is adopted. Thus Elements K are found. The representation of the normals from micrometer observations is —1″ to +2″ in the plane.

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TABLE 13.—*Elements—(447) Valentine (1899 ES)*

Letter	Epoch	M. T.	M.	ω	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A......	1899 Dec. 3.5.....	Berlin............	4 21 32	319 16 21	72 18 33	4 49 33
B......	1899 Dec. 5.5.....	Berlin............	4 40 42	319 15 3	72 24 5	4 49 4
	Osc. Nov. 5.0.....					
C......	1901 Feb. 8.0.....	Berlin............	87 2 32	318 53 31	72 24 0	4 49 4
D......	1902 Apr. 4.0.....	Berlin............	167 42 14	318 29 20	72 23 32	4 49 4
E......	1904 Oct. 10.0.....	Berlin............	345 41 51	316 22 10	72 19 44	4 49 5
F......	1906 Feb. 2.0.....	Berlin............	79 55 40	313 38 14	72 25 5	4 49 21
G......	1899 Nov. 5.0.....	Berlin............	358 57 25	319 13 45	72 24 10	4 49 4
H......	1899 Nov. 5.0.....	Berlin............	2 42 27	315 27 43	72 25 38	4 49 9
I......	1899 Nov. 5.0.....	Berlin............	358 52 18	319 13 42	72 24 16	4 49 4
J......	1899 Nov. 5.0.....	Berlin............	358 52 21	319 13 42	72 24 11	4 49 3
K......	1899 Nov. 5.0.....	Berlin............	358 57 21	319 13 40	72 24 17	4 49 4

Letter	Epoch	M. T.	φ	μ	Equinox	Authority
			° ′ ″	″		
A......	1899 Dec. 3.5.....	Berlin............	2 36 37	687.012	1900.0	H. Kreuts
B......	1899 Dec. 5.5.....	Berlin............	2 34 33	687.5969	1900.0	H. Kreuts
	Osc. Nov. 5.0.....					
C......	1901 Feb. 8.0.....	Berlin............	2 35 5	687.6846	1900.0	H. Kreuts
D......	1902 Apr. 4.0.....	Berlin............	2 36 17	688.1604	1900.0	H. Kreuts
E......	1904 Oct. 10.0.....	Berlin............	2 40 15	686.5435	1900.0	H. Kreuts
F......	1906 Feb. 2.0.....	Berlin............	2 38 34	687.9066	1910.0	H. Kreuts
G......	1899 Nov. 5.0.....	Berlin............	2 34 32	687.3550	1900.0	Osten*
H......	1899 Nov. 5.0.....	Berlin............	2 25 42	687.3504	1900.0	Osten
I......	1899 Nov. 5.0.....	Berlin............	2 34 32	687.6016	1900.0	Osten
J......	1899 Nov. 5.0.....	Berlin............	2 34 32	687.6018	1900.0	Osten
K......	1899 Nov. 5.0.....	Berlin............	2 34 32	687.3884	1900.0	Osten

* Mean elements.

(588) ACHILLES, 1906 TG.

Discovered by Wolff[1] at Heidelberg 1906, February 22.

From observations of February 22, and March 5, Berberich[2] computed Elements A for a circular orbit (search ephemeris for April 1906).

The first elliptic Elements B were published by Berberich.[3] They are based on observations 1906, February 22, March 23, and April 22. An ephemeris for May and June, 1906, is also included. He also points out that the aphelion point lies far beyond Jupiter's orbit, and that the present orbit has had its form and position for a long time.

From Elements B, Charlier[4] finds that Achilles is approximately 55° ahead of Jupiter, consequently very close to one of Lagrange's libration points. He also points out that we may have here a case (as he has shown)[5] where the planet does not remain at the apex of the equilateral triangle, as suggested by Lagrange, but oscillates about it with a period of about 148 years.

Comments on the character of the orbit of Achilles have been published by Berberich,[6] Crommelin,[7] Ristenpart,[8] Stroobant.[9]

On the basis of observations extending from 1906, February 22, to May 19, Bidschof[10] has published a set of Elements C and an ephemeris for 1907. A continuation of the ephemeris with corrections to the same was published by Bidschof.[11] An observation 1907, February 12, gives $\Delta\alpha$ —51^s, and $\Delta\delta$ +6′.7.

An ephemeris for 1909 based on Elements B. J. 1911 with special perturbations due to Jupiter was published by Franz.[12]

For the following years, to 1919, the B. J. (Kleine Planeten) publishes elements by Bidschof brought forward to a new epoch and mean equinox,[13] Elements D.

As an application of Leuschner's[14] satellite method, Einarsson[15] computed a preliminary orbit, (Elements E), based on observations 1907, February 12, April 15, and June 2. Special perturbations, due to Jupiter, computed by Encke's method, were included for the period covering the observations. The maximum residuals, for nine observations of 1907, were $\Delta\alpha \cos\delta$ —3″.6, $\Delta\delta$ ±0″.7. Elements E were used, without perturbations, to represent an observation 1906, May 19, with the following results: $\Delta\alpha \cos\delta$ +1′01″.8, $\Delta\delta$ —39″.2.

The most recent work on Achilles was done by Julie M. Vinter-Hansen.[16] With Bidschof's Elements D, the special perturbations, due to Jupiter and Saturn, were computed from 1906 to 1914. All observations of 1906 and 1907, two of 1913, and one of 1914, were then represented and residuals determined. With these as a basis, eleven normal places were formed, weighted according to number of observations in each.

The residuals, (freed from perturbations), for the 1906 and 1907 normal places are small. For normal place X, (1913), $\Delta\alpha \cos \delta$ $+2781''.2$, $\Delta\delta$ $+1685''.0$; for normal place XI, (1914), $\Delta\alpha \cos \delta$ $+1664''.1$, $\Delta\delta + 852''.7$. The resulting orbit improvement gave Elements F. The residuals for the normal places from Elements F vary from $+19''.8$ to $-40''.9$ in $\Delta\alpha \cos \delta$, and $-9''.2$ to $+23''8$ in $\Delta\delta$. With Elements F the special perturbations were recomputed by J. Braae. With Elements F and the new perturbations, new residuals for the normal places were determined and the orbit improved on this basis. When the corrections to the elements were put back into the equations of condition, the residuals were of the same order as prior to the solution. Miss Hansen concludes a normal place must be incorrect. The equations of condition were again solved, with the last normal place omitted. The resulting Elements G represent the ten normal places in $\Delta\alpha$ cos δ $-1''.5$ to $+5''.8$ and $\Delta\delta$ $-1''.3$ to $+2''.3$.

The normal place XI gives residuals $\Delta\alpha \cos \delta$ $-1817''.2$, $\Delta\delta$ $-1172''.0$, which indicates that it does not belong to Achilles.

The special perturbations due to Jupiter for 1915 to 1920 are published by Miss Hansen.[17] They were computed from Elements G. These elements are published and utilized in Kleine Planeten since 1920. In Ark. för Math. Bd. 4 Nr. 20 Linders developed the approximate theory for planets near the libration points and gives the principal perturbations of the elements of 588. Cf. Heinrich V. J. S. 1913.

REFERENCES

[1] A. N. vol. 170, p. 353.
[2] A. N. vol. 171, p. 11.
[3] A. N. vol. 171, p. 127.
[4] A. N. vol. 171, p. 213.
[5] Meddelanden fran Lunds Observatorium, No. 18.
[6] Nat. Rund. vol. 21, p. 485–486.
[7] Observatory. vol. 29, p. 352–355. Pop. Ast. vol. 14, p. 472–475.
[8] H. u. E. vol. 18, p. 517-521.
[9] Ciel et Terre. vol. 27, p. 161–164.
[10] A. N. vol. 174, p. 45–48.
[11] A. N. vol. 174, p. 175.
[12] A. N. vol. 180, p. 295.
[13] B. J. 1914, p. 30.
[14] L. O. Pub. vol. vii, p. 455.
[15] In Manuscript (Berkeley).
[16] Pub. og mindre Meddelelser fra Köbenhavns Obs. No. 37.
[17] A. N. vol. 208, p. 345.

TABLE 14.—*Elements—(588) Achilles (1906 TG.)*

Letter	Epoch	M. T.	M.	ω	Ω	i
A......	1906 Mar. 5.5...	Berlin......	(u) 186° 15.′9	....	316° 1.′2	11° 43.′2
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
B......	1906 Feb. 22.5....	Berlin......	48 57 24	120 25 50	315 34 7	10 20 53
C......	1906 Feb. 22.5....	Berlin......	43 45 37	{129 24 11 129 24 10	315 31 7 315 31 58	10 16 36 10 16 36
D......	1907 Apr. 15.5....	Berlin......	80 18 12	125 37 50	315 36 2	10 18 25
E......	1907 Apr. 15.66...	Greenwich..	80 23 50	125 25 21	315 35 45	10 18 45
F......	1907 May 28.0....	Berlin......	82 54 47	127 7 10	315 34 26	10 17 53
G......	1907 May 28.0....	Berlin......	84 3 2	125 36 22	315 35 59	10 18 14

TABLE 14—*Elements*—(*588*) *Achilles* (*1906 TG.*)—Continued

Letter	Epoch	M. T.	φ	μ	Equinox	Authority
			° ′ ″	″		
A......	1906 Mar. 5.5.......	Berlin........	....	312.32	1906.0	Berberich
B......	1906 Feb. 22.5........	Berlin........	9 38 43	295.133	1906.0	Berberich
C......	1906 Feb. 22.5........	Berlin........	8 10 15	294.703	{1906.0 1907.0	Bidschof
D......	1907 Apr. 15.5........	Berlin........	8 42 54	295.464	1910.0	Bidschof
E......	1907 Apr. 15.66.......	Greenwich....	8 45 49	295.6847	1907.0	Einarsson
F......	1907 May 28.0........	Berlin........	8 25 19	294.71497	1910.0	Miss Hansen
G......	1907 May 28.0........	Berlin........	8 36 43	295.96333	1910.0	Miss Hansen

(617) PATROCLUS, 1906 VY.

Discovered by Kopff[1] at Heidelberg, October 17, 1906.

Preliminary Elements A were computed by Heinrich[2] based on four observations October 21 to December 7.

From Elements A Charlier[3] has noted that the longitude of Patroclus is approximately 60° behind Jupiter as compared with Achilles and Hector which are approximately 60° ahead of Jupiter. From the same elements, Strömgren[3] has made a similar comparison.

A second set of preliminary Elements B were computed by Heinrich[4] by variation of the distances on the basis of five observations from 1906 October 21 to December 7. On page 339 of the same reference the correction is made that Elements B refer to equinox 1906. Elements B are used to compute the special perturbations, by the method of variation of the constants, and a third set of Elements C with an ephemeris for 1907 is published.[5] A continuation of the ephemeris is published on page 251 of the same reference. Observations November 8 and 10, 1907, show the following corrections to the ephemeris: $\Delta\alpha = -34^s$ $\Delta\delta = -0'.8$.

Ephemerides for the oppositions 1908, 1909, 1910, 1912, 1913, by Heinrich[6] are based on Elements C. The correction to the ephemeris in 1909 was $\Delta\alpha - 39^s$ and $\Delta\delta + 3'.3$.

From 1914 to 1918 the ephemerides based on Elements C are published in Kleine Planeten.

An extensive application of Wilkens' method[7] for planets of Jupiter's group, is made by Drucker.[8] He first takes Heinrich's Elements C as a basis for the computation of the perturbations due to Jupiter and Saturn, by the method of variation of elements. Then corrects Elements C on the basis of twelve normal places (1906 to 1918). See Elements D. The residuals for the twelve normal places, from Elements D plus the special perturbations due to Jupiter and Saturn, vary from $-7''.9$ to $+10''.3$ for $\Delta\alpha \cos\delta$ and $-4''.8$ to $+4''.9$ for $\Delta\delta$.

With Elements D the perturbations due to Jupiter and Saturn are again computed and with the former twelve normal places and a new thirteenth normal place, (1919), the Elements D were corrected to a new set, E. With this new set of Elements E, and the new perturbations, the residuals for the normal places vary from $-3''.4$ to $+3''.5$ for $\Delta\alpha \cos\delta$ and from $-2''.8$ to $+2''.9$ for $\Delta\delta$. He states these Elements E may be considered as definite. In a footnote on page 27 of the same reference, the residuals for an observation 1920, December 16, are $\Delta\alpha \cos\delta$ $+4''.4$ and $\Delta\delta$ $+1''.4$. The ephemeris for 1920 was computed from Elements E, plus special perturbations due to Jupiter and Saturn.

Then he derives corresponding elements in Wilkens' system, Elements F and G. Elements F correspond to Elements D, and Elements G correspond to Elements E. He states that it was only necessary according to Wilkens' method to compute Elements F.

He now computes positions (α and δ), for the thirteen normal places, (1) from Elements E, with special perturbations due to Jupiter; (2), the same places from Elements E without perturbations; (3), the same places from Elements G without perturbations; (4), the same places from Elements E with special perturbations due to Jupiter and Saturn. A comparison is then made between the positions computed by the various systems, (1), (2), (3), (4). The table of differences shows that Saturn's perturbations do not become effective until 1918. It shows, as we should expect, that for the period of osculation, the various systems give the same results. It finally shows that positions computed from (3) come closer to those computed by (1), than the places computed by (2).

Finally he computes the special perturbations due to Jupiter, by the variation of elements, with the aid of Elements G and shows that they are of the same order as those computed with Elements E.

The ephemerides published in Kleine Planeten for 1920 and 1921 are based on Elements E, plus special perturbations due to Saturn and Jupiter.

An ephemeris for the opposition using Elements E, 1918-1919, was computed by Paul Maitre.[9] Ephemerides for oppositions 1919-1920, 1920-1921, and 1922, are also published by Drucker.[10]

REFERENCES

[1] A. N. vol. 172, p. 387.
[2] A. N. vol. 175, p. 87.
[3] A. N. vol. 175, p. 89.
[4] A. N. vol. 175, p. 291.
[5] A. N. vol. 176, p. 193.
[6] A. N. vol. 179, p. 223—vol. 180, p. 45—vol. 183, p. 207—vol. 190, p. 395—vol. 194, p. 208.
[7] A. N. vol. 205, No. 4906.
[8] A. N. vol. 214, No. 5114.
[9] Cir. O. M. No. 85, 1918.
[10] B.-Z. der A. N. No. 14, 1919—No. 48, 1920—No. 1, 1922.

TABLE 15.—*Elements (617)—Patroclus (1906 VY)*

Letter	Epoch	M.T.	M	ω	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A......	1906 Oct. 21.5..........	Berlin......	41 31 55	297 28 37	43 21 39	22 16 47
B......	1906 Nov. 29.0..........	Berlin......	41 27 30	302 11 27	43 25 32	22 3 33
C......	1907 Dec. 14.0..........	Berlin......	73 1 25	302 25 48	43 28 36	22 3 15
D......	1906 Nov. 29.0..........	Berlin......	42 00 14	−58 26 46	43 27 49	22 7 00
E......	1906 Nov. 29.0..........	Berlin......	41 59 24	−58 25 39	43 27 49	22 6 57
F......	1906 Nov. 29.0..........	Berlin......	42 19 5	−58 45 27	43 27 49	22 7 00
G......	1906 Nov. 29.0..........	Berlin......	42 18 15	−58 44 21	43 27 49	22 6 57

TABLE 15—*Elements*—(617) *Patroclus* (*1906 VY.*)—Continued

Letter	Epoch	M.T.	φ	μ	Equinox	Authority
			° ′	″		
A......	1906 Oct. 21.5............	Berlin......	8 42 41	300.145	1906.0	Heinrich
B......	1906 Nov. 29.0............	Berlin......	8 16 7	300.659	1906.0	Heinrich
C......	1907 Dec. 14.0............	Berlin......	8 14 38	300.532	1910.0	Heinrich
D......	1906 Nov. 29.0............	Berlin......	8 15 33	300.7805	1910.0	Drucker
E......	1906 Nov. 29.0............	Berlin......	8 15 32	300.7654	1910.0	Drucker
F......	1906 Nov. 29.0............	Berlin......	8 13 9	301.4459	1910.0	Drucker
G......	1906 Nov. 29.0............	Berlin......	8 13 9	301.4309	1910.0	Drucker

(624) HECTOR, 1907 XM.

Discovered by Kopff[1] at Heidelberg on February 10, 1907.

A preliminary orbit, Elements A, was computed by Strömgren[2] based on observations from February 10 to April 16. These elements gave the following residuals:

1907	$\Delta\lambda$	$\Delta\beta$
Feb. 10	$-1''.0$	$+2''.2$
April 19	$+2\ .8$	$+8\ .3$

He reports that Hector is another planet with mean motion nearly equal to that of Jupiter. Since the perturbations of Jupiter are small and the perturbations of the other planets will be ineffective for a long time, this planet will remain in the neighbourhood of the libration point for a long time.

An observation by Palisa February 29, 1908, gives a correction to the ephemeris[3] based on Elements A as follows: $\Delta\alpha = -37^s$ $\Delta\delta = +6'.3$. An ephemeris is published by Strömgren[4] for the opposition of 1909 based on Elements A.

In preparation for the ephemeris of 1911, Strömgren,[5] assisted by J. Fischer-Petersen, computes a new set of Elements B, based on nine normal places (oppositions 1907, 1908, 1909), taking into account the perturbations due to Jupiter and Saturn. These elements represent the normal places, $\Delta\alpha\cos\delta$ between $+0'.30$ and $-.17$, and $\Delta\delta$ between $-0'.04$ and $+0'.16$.

The planet was next observed in July 1911. The comparison between observation and ephemeris (from Elements B) was unsatisfactory. Strömgren, assisted by Ruben Andersen,[6] again investigated the orbit based on observations from 1907 to 1911. For this purpose Elements A were used to compute the residuals for five observations, July 4 to 16, 1911. A tenth normal place was formed from these five places with the following residuals $\Delta\alpha\cos\delta = -52'\ 23''.9$ $\Delta\delta = -11'\ 36''.6$. A weight of five (number of observations) was given to this normal place. Combining this normal place with the solution that led to Elements B, a new set of elements was derived, Elements C. With these elements the special perturbations, due to Jupiter and Saturn, were computed by the method of Encke, for 1906 to 1912. The observations for 1907 and 1908 were then computed without taking into account the perturbations (reported as being small) and the observations of 1909 and 1911 were represented with perturbations. From these representations ten new normal places were formed and new Elements D were obtained from a least square solution. This set represents the normal places, $\Delta\alpha\cos\delta$ between $-7''.1$ and $+3''.8$, and $\Delta\delta$ between $-1''.6$ and $+3''.0$.

With Elements D and special perturbations, the maximum residuals for opposition of 1912 were $\Delta\alpha\cos\delta = +2^s.12$ and $\Delta\delta = +26''.6$. Strömgren, assisted by Julie M. Vinter Hansen,[7] again investigated the orbit of Hektor on the basis of thirteen normal places (1907 to 1912). A least square solution led to Elements E. At the conclusion of this work,[7] osculating elements were computed for the epoch of each year, 1907 to 1913, based on Elements E and the special perturbations published in the same reference. The ephemeris for 1913 was published in A. N., vol. 198, p. 367.

In A. N., vol. 200, pp. 79–82, Strömgren publishes the comparisons between the observations and the ephemeris, which was based on Elements E and the perturbations due to Jupiter and Saturn, for 1913 and 1914. The maximum residual for 1913 for $\Delta\alpha\cos\delta = +0^s.53$ and for $\Delta\delta = +7''.1$. For 1914 the maximum residual for $\Delta\alpha\cos\delta = +0^s.51$ and $\Delta\delta = +11''.6$. Strömgren regards these residuals as satisfactory.

As an application of Leuschner's satellite method, S. Einarsson[8] computed a preliminary orbit of Hector based on observations 1907, February 10, March 11, and April 16, Elements F. These elements represented an observation of May 2, 1908, as follows: $\Delta\alpha\cos\delta = -4'\,32''.4$ $\Delta\delta = +3'\,31''.9$. With Elements F, the special perturbations by Encke's method were computed for the 1907 opposition. New elements were then computed from observations 1907, February 10, March 11, and a normal place from observations April 12, 16, and 19, by a differential correction of Elements F. These new Elements G represented the opposition of 1908, May 2, as follows: $\Delta\alpha + 30''$, $\Delta\delta + 95''$, and for 1909, April 17, $\Delta\alpha + 6'.1$; $\Delta\delta - 6'.7$. The application of Leuschner's method thus yielded far better results than the ordinary method followed by Strömgren: 1908, $\Delta\alpha = -9'\,15''$ $\Delta\delta = +6'\,18''$. No further work was done on this planet by Einarsson since Strömgren and his colleagues had already made extensive investigations.

An ephemeris for 1915 is published by J. Fischer-Petersen,[9] using Elements E and taking into account perturbations due to Jupiter and Saturn.

The elements and ephemeris for 1916[10] are based on Strömgren's Elements E with special perturbations by Jupiter and Saturn brought forward.

Elements and ephemerides are given in Kleine Planeten for each year to 1921. No further comment is published regarding the elements.

An ephemeris is published by M. Henri Blondel[11] for the opposition of 1921 which has been corrected on the basis of an observation at

Algiers[12] on May 6, 1921. This observation gave a correction to the ephemeris published in Kleine Planeten of $+7^m.0$ in right ascension and —80′ in declination.

In A. N., vol. 215, p. 249, A. Wilkens has published an article, "Uber die Säkularen Veränderungen der Grossen Achsen der Bahnen der Planeten der Jupiter Gruppe."

In A. N., vol. 175, p. 89, Charlier gives a brief discussion on the orbits of the Trojan group, regarding their motion about the libration points.

In A. N., vol. 206, p. 235, A. Koref outlines his investigation regarding the motion of Hector. His preliminary work is based on eighteen normal places (1907 to 1914). The investigation will be completed when observations of 1918 and 1919 are available.

REFERENCES

[1] A. N. vol. 174, p. 63.
[2] A. N. vol. 175, p. 14.
[3] A. N. vol. 177, p. 123.
[4] A. N. vol. 180, p. 327.
[5] Publikationer og mindre Meddelelser fra Köbenhavns Observatorium No. 6. A. N. vol. 188, p. 395.
[6] Publikationer og mindre Meddelelser fra Köbenhavns Observatorium No. 8.
[7] Publikationer og mindre Meddelelser fra Köbenhavns Observatorium No. 12.
[8] In manuscript (Berkeley).
[9] A. N. vol. 201, p. 335. B. A. J. 1917.
[10] Eph. der Kleinen Planeten, 1916, p. 17, p. 77, p. 93, p. 97.
[11] Cir. O. M. No. 480 (1921).
[12] Cir. O. M. No. 171, second series.

TABLE 16.—*Elements—(624) Hector (1907 XM)*

Letter	Epoch	M.T.	M	ω	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A......	1907 Feb. 10.0.........	Berlin......	335 47 12	183 51 52	341 58 25	18 7 17
B......	1907 Feb. 10.0.........	Berlin......	343 51 48	175 6 42	341 58 57	18 8 34
C......	1907 Feb. 10.0.........	Berlin......	345 38 38	173 5 26	341 59 47	18 9 13
D......	1907 Feb 10.0.........	Berlin......	343 40 12	175 19 0	341 59 18	18 8 50
E......	1907 Feb. 10.0.........	Berlin......	343 48 55	175 9 30	341 59 15	18 8 45
F......	1907 Mar. 11.36........	Greenwich..	341 56 43	179 46 25	341 57 32	18 8 05
G......	1907 Mar. 11.36........	Greenwich..	348 27 23	172 44 30	341 57 14	18 8 19

Letter	Epoch	M.T.	φ	μ	Equinox	Authority
			° ′ ″			
A......	1907 Feb. 10.0..........	Berlin......	2 8 24	292.584	1907.0	Strömgren
B......	1907 Feb. 10 0..........	Berlin......	56 46	293.1585	1910.0	Strömgren
C......	1907 Feb. 10.0..........	Berlin... ..	1 43 45	295.3661	1910.0	Strömgren
D......	1907 Feb. 10.0..........	Berlin......	1 56 52	293.1072	1910.0	Strömgren
E......	1907 Feb. 10.0..........	Berlin......	1 56 29	293.1782	1910.0	Strömgren
F......	1907 Mar. 11.36.........	Greenwich..	2 03 07	292.7487	1910.0	Einarsson
G......	1907 Mar. 11.36.........	Greenwich..	1 55 32	293.1164	1910.0	Einarsson

(659) NESTOR, 1908 CS.

Discovered by Wolf[1] at Heidelberg, 1908, March 23.

From observations of 1908, March 25 and May 2, Ebell[2] computed a circular orbit. From these Elements A it is evident that the planet belongs to the Trojan group, Achilles type, and he notes the planet's position is near a libration point (60° ahead of Jupiter).

Preliminary elliptic Elements B, with an ephemeris for 1908, were computed by Ebell[3] from observations 1908, March 23, April 26, and May 19. With Elements B Ebell[4] publishes an ephemeris for 1909.

The improvement of Elements B was undertaken by Anderson.[5] Special perturbations due to Jupiter and Saturn were computed with Elements B. Six normal places were formed from eleven observations extending from 1908, March 23 to 1912, September 9.

The coefficients for the differential equations were computed by the method given in Oppolzer.[6] The solution of the equations gave abnormal corrections when both or either normal places IV and V were used. (IV and V are single observations). His final solution was based on the first four normal places, (1908 to 1909). The resulting Elements C were used to compute the special perturbations due to Jupiter and Saturn and an ephemeris for 1913.

Wolf reports [7] that Nestor cannot be found at ephemeris position.

An ephemeris for 1914 based on Elements C, plus perturbations, (disturbing planets not stated), was published by Andersen.[8] He states no other observations for this planet are available. Reported observations since 1909 do not appear to belong to Nestor.[9]

An ephemeris for 1915 based on Elements C by Anderson, was published by Strömgren.[10] He states the correction to ephemeris, based on an observation of 1914, December 20, is $\Delta\alpha$ $+62^s$, $\Delta\delta$ $-7'.6$.

Another attempt to improve the orbit of Nestor was made by Andersen[11] in 1917. In this attempt six normal places were formed from observations in 1908, 1909, and 1917. The perturbations due to Jupiter and Saturn were computed from Elements C, and the normal places represented from the same elements. The starting residuals for observation, 1908 and 1909, were small, (maximum $-2''.4$); for the normal place V, (two observations of 1917), $\Delta\alpha \cos \delta$ $-2008''.8$, $\Delta\delta$ $+458''.4$; for the normal place VI, (two observations of 1917), $\Delta\alpha$ $-1934''.7$, $\Delta\delta$ $+392''.2$. The differential corrections were computed as in the earlier work.[6] For resulting elements see D. These elements left the following residuals for the V and VI normal places:

	$\Delta\alpha \cos \delta$	$\Delta\delta$
V	$-4''.7$	$-6''.2$
VI	$+5.0$	$+7.2$

Elements D are sufficiently accurate to show that observations reported for this planet in 1914-1915 do not belong to Nestor.

With Elements D and perturbations computed from Elements C an ephemeris was computed for 1918.

Julie M. Vinter-Hansen has published[12] the special perturbations, (disturbing planets not stated), computed by Pedersen from 1907 to 1918. Also Andersen's Elements D brought up to epoch of 1918, (Elements E), and an ephemeris for 1919.

Seagrave has published[13] an ephemeris for 1920. No reference is made as to what elements were used.

The most recent work on Nestor is by Kristensen.[14] He begins with Andersen's Elements D, and computes the special perturbations due to Jupiter and Saturn and then represents all the observations available from 1908 to 1919. The maximum initial residuals (perturbations included), for the 1919 observations are $\Delta\alpha \cos\delta$ $-64^s.29$, $\Delta\delta$ $+494''.8$. He then forms eleven normal places based on all the available observations. The maximum initial residuals appear in the IX Normal place, $\Delta\alpha \cos\delta$ $-963''.0$, $\Delta\delta$ $+494''.0$, (perturbations included). The resulting solution of the eleven equations gives Elements F. These elements represent the normal places satisfactorily. The maximum residuals are $\Delta\alpha \cos\delta$ $+0^s.16$, $\Delta\delta$ $+7''.2$. He states these Elements F may be considered as a definitive system.

The elements published in Kleine Planeten, 1917 to 1920, are those of Andersen (Elements C), brought up to new epochs.

REFERENCES

[1] A. N. vol. 177, p. 287.
[2] A. N. vol. 177, p. 399.
[3] A. N. vol. 178, p. 71.
[4] A. N. vol. 180, p. 213.
[5] A. N. vol. 195, No. 4678, p. 433.
[6] Oppolzer. vol. ii, p. 390–391.
[7] A. N. vol. 196, p. 14.
[8] A. N. vol. 199, p. 221.
[9] Veröff. R. I. No. 42, remarks 5 and 15.
[10] A. N. vol. 200, p. 56.
[11] A. N. vol. 206, No. 4923, p. 17.
[12] A. N. vol. 208, p. 15–16.
[13] A. J. vol. 32, p. 167.
[14] Pub. og mindre Meddelelser fra Köbenhavns Obs. No. 37.

TABLE 17.—*Elements (659)—Nestor (1908 CS.)*

Letter	Epoch	M. T.	M.	ω	Ω	i
A.........	1908 Mar. 25.5...	Berlin.....	(u) 196° 55'.8	....	350° 55'.7	4° 40'.1
			° ' "	° ' "	° ' "	° ' "
B.........	1908 Mar. 23.5...	Berlin.....	240 38 5	327 31 28	349 57 42	4 31 15
C......... Osculation...	1908 Mar. 23.5 1908 Apr. 12.0 } ..	Berlin.....	240 3 56	328 4 54	350 0 1	4 31 31
D......... Osculation...	1908 Mar. 23.5 1908 Apr. 12.0 } ..	Berlin.....	237 29 22	329 41 1	350 2 4	4 31 47
E.........	1918 Mar. 11.0...	Greenwich..	179 57 36	331 58 33	350 5 22	4 31 45
F......... Osculation...	1908 Mar. 23.5 1908 Apr. 12.0 } ..	Berlin.....	238 18 00	329 7 41	350 1 29	4 31 44

TABLE 17.—*Elements* (*659*)—*Nestor* (*1908 CS.*)—Continued

Letter	Epoch	M. T.	φ	μ	Equinox	Authority
			° ′ ″	″		
A.........	1908 Mar. 25.5...	Berlin.....	....	290.67	1908.0	Ebell
B.........	1908 Mar. 23.5...	Berlin.....	6 23 59	300.785	1908.0	Ebell
C......... Osculation...	1908 Mar. 23.5 1908 Apr. 12.0 } ..	Berlin.....	6 26 44	301.0002	1910.0	R. Andersen
D......... Osculation...	1908 Mar. 23.5 1908 Apr. 12.0 } ..	Berlin.....	6 1 11	301.5134	1910.0	R. Andersen
E.........	1918 Mar. 11.0...	Greenwich..	6 2 00	299.803	1919.0	J. M. V. Hansen
F......... Osculation...	1908 Mar. 23.5 1809 Apr. 12.0 } ..	Berlin.....	6 7 13	301.2194	1910.0	Kristensen

(716) BERKELEY, 1911 MD.

Discovered by Palisa[1] at Vienna on July 30, 1911.

The first preliminary Elements A were computed by Hopfner[2] and are based on observations 1911, July 30, August 17, and September 3. He also gives an approximate ephemeris for 1911.

Hopfner's Elements A represented the observation of 1906, July 16, for (1906 UN^b) as follows: $\Delta\alpha$ $-1^m.4$ $\Delta\delta$ $-4'.8$. Berberich[3] announces (1906 UN^b) is identical with (716) Berkeley.

An ephemeris is published by Cohn[4] for 1912. An observation by Palisa[5] for November 4, 1912, gives a correction to the ephemeris, $\Delta\alpha$ $-2^m.7$ and $\Delta\delta$ $-8'$.

A new set of Elements B computed by Stracke[6] is based on observations 1911, August 3, 29, and September 23. These elements represent the observation of 1906, July 16, for (1906 UN^b) as follows: $\Delta\alpha$ $+2^m.4$ and $\Delta\delta$ $+4'$.

Stracke's elements are published and used in B. J. 1915 and up to 1919 in Kleine Planeten.

An orbit, Elements C, based on three normal places 1911, July 30, August 22, Septemer 17, was computed by Neubauer[7] using Leuschner's Short Method.[8] A comparison between observation and computation, for observations in 1906, 1912, 1914, of Elements C and Stracke's Elements B, showed that Elements C gave the better representation. Neubauer also computes general perturbations by the Bohlin method,[9] with the tables computed by Wilson[10] for the group 750.[11] More accurate starting elements connecting several oppositions by special perturbations seem to be desirable.

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[1] A. N. 189, p. 109.
[2] A. N. 189, p. 244.
[3] A. N. 189, p. 364.
[4] A. N. 192, p. 426.
[5] A. N. 193, p. 62.
[6] A. N. 192, p. 421–423.
[7] L. O. Bull. No. 301.
[8] L. O. Pub. vol. vii, part viii.
[9] "Formeln and Tafeln sur gruppenweise Berechnung der Allgemeinen Störungen Benachbarter Planeten," Upsala 1896, and "Sur le Dévelopement des Perturbations Planétaires," Stockholm 1902.
[10] Ast. Iakttagelser Och Undersökningar, Band 10, No. 1, Stockholm.

TABLE 18.—*Elements (716)—Berkeley (1911 MD)*

Letter	Epoch	M.T.	M	ω	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A......	1911 Aug. 17.5..........	Berlin......	120 23 1	46 3 5	146 51 41	8 37 59
B......	1911 Aug. 18.5..........	Berlin......	118 6 10	48 49 6	146 57 7	8 27 43
C......	1911 Aug. 22.4..........	Berlin......	118 25 50	48 49 15	146 54 49	8 23 33

TABLE 18.—*Elements (716)—Berkeley (1911 MD)*.—Continued

Letter	Epoch	M.T.	φ	μ	Equinox	Authority
			° ′ ″	″		
A......	1911 Aug. 17.5...........	Berlin......	5 28 46	753.940	1911.0	Hopfner
B......	1911 Aug. 18.5...........	Berlin......	5 5 17	754.565	1911.0	Stracke
C......	1911 Aug. 22.4...........	Berlin......	5 23 38	753.7233	1911.0	Neubauer

(718) ERIDA, 1911 MS.

Discovered by Palisa[1] at Vienna, 1911, September 29.

Preliminary Elements A, and an ephemeris based on observations 1911, September 29, October 13, and October 28, are published by Cohn.[2] These elements were used by the B. J. 1915 and 1916.

New Elements B were obtained by Strehlow[3] from observations 1914, February 28, March 18, and March 29. The method of the variation of the distances was utilized so that observations of 1911 and 1914 were well represented.

A correction of $+1''.5$ to μ was applied in order to represent observations of 1904 for (1904, OD) which is supposed to be identical to (718) Erida.

As a test of Leuschner's Short Method for computing orbits,[4] Mundt[5] has published two sets of elements C and D, which are based on the observations used by Cohn.[2]

For the purpose of comparison, ephemeris places were computed from Cohn's Elements A, Strehlow's Elements B, and Mundt's Elements C.

1917 G. M. T.	α	δ	
Oct. 22.5	2^h56^m3	$+17° 09'$	Elements C.
Oct. 22.5	2 55.9	+17 07	Elements B.
Oct. 22.5	2 53.5	+16 56	Elements A.
Dec. 1.5	2 25.3	+15 39	Elements C.
Dec. 1.5	2 25.1	+15 36	Elements B.
Dec. 1.5	2 22.9	+15 24	Elements A.

No comparison with observations in 1917 has been made.

Strehlow's Elements B have been published and utilized by B. J. (Kleine Planeten) since 1915. An observation by Palisa[6] on 1919, January 6, gave corrections to the ephemeris as follows: $\Delta\alpha -1^m.7$ $\Delta\delta -3'$.

The object of Mundt's work was to test the possibility of deriving by properly chosen methods as satisfactory elements from a single opposition as are ordinarily obtained from at least two oppositions. This result was realized in this case. His work was duplicated by Miss Easton[5] on a slightly different plan of removing the residuals. Elements D.

REFERENCES

[1] A. N. vol. 189, p. 295.
[2] A. N. vol. 192, p. 421–423.
[3] B. J. 1917, p. 36 and 106.
[4] L. O. Pub. vol. vii.
[5] L. O. Bull. No. 302.
[6] E. Z. der A. N. 1919, No. 561.

TABLE 19.—*Elements (718) Erida (1911 MS)*

Letter	Epoch	M.T.	M	ω	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A......	1911 Sept. 29.5..........	Berlin......	149 0 40	169 56 47	39 22 47	7 3 55
B......	1914 Apr. 1.5..........	Berlin......	320 18 15	168 8 30	39 44 16	6 58 13
C......	1911 Oct. 13.4..........	Berlin......	156 34 10	166 36 12	39 42 41	6 59 5
D......	1911 Oct. 13.4..........	Berlin......	156 03 25	166 55 59	39 43 51	6 58 49

Letter	Epoch	M.T.	φ	μ	Equinox	Authority
			° ′ ″	″		
A......	1911 Sept. 29.5..........	Berlin....	12 5 35	664.65	1911.0	F. Cohn
B......	1914 Apr. 1.5..........	Berlin....	11 28 39	664.412	1910.0	Strehlow
C......	1911 Oct. 13.4..........	Berlin....	11 19 7	663.865	1911.0	C. Mundt
D......	1911 Oct. 13.4..........	Berlin....	11 20 11	663.769	1911.0	Miss E. J. Easton

(884) PRIAMUS, 1917 CQ.

Discovered by Wolf[1] at Heidelberg on September 22, 1917.

Wilkens outlines[2] his preliminary investigation regarding the motion of this fifth member of the Trojan group. For this purpose he utilizes preliminary Elements A, by Berberich. The libration point is 60° behind Jupiter and the planet oscillates about this point in approximately 150 years. He states that μ varies between 294″.27 and 303″.99. He also points out that his succeeding work will show that Priamus and Patroclus are diametrically opposite in the small libration ellipse.

Klose applies[3] Wilkens' method[4] for taking into account the principal perturbations of Jupiter, to Priamus. In this method the principal perturbations by Jupiter are accounted for by centering in the Sun the mass of the Sun-Jupiter system. For this study Klose utilizes Berberich's Elements A, bringing them up to mean equinox 1925.0, Elements B, and compares the coordinates computed from Elements B plus special perturbations with those computed from Elements C, which were derived by Wilkens' method. The difference in the representation by the two sets of elements for observations from 1917 October 14 to 1918 December 28, did not exceed $0^s.5$ in right ascension and 2″ in declination. Klose concludes that Wilkens' method is applicable beyond this short period for immediate Ephemeris purposes.

As an example of his method[5] of integrating the differential equations for the perturbations in the coordinates for planets of the Jupiter group, Wilkens[6] gives a numerical application for Priamus. His results are similar to those of Klose. Klose[7] publishes a further comparison between the usual method of special perturbations due to Jupiter and Wilkens' method. For this purpose he compares results gotten from Elements B and a new set of Elements D, derived by Wilkens' method. He then brings up Elements B with perturbations due to Jupiter up to epoch 1918 (Elements E), and Elements D are brought forward to epoch 1918 by Wilkens' method. An ephemeris is published for the opposition 1919 for which the perturbations of Saturn are also taken into account.

In an article on "Bemerkenswerte Eigenschaften der Bahnen der Planeten der Jupitergruppe," Wilkens[8] points out that the ascending nodes of the six Trojan planets lie with one exception, (Patroclus), in the same quadrant. He also forms a mean value of the ascending nodes from certain planets of the group and compares the individual values with these mean values. He also refers to his article[9] regarding the time of maximum and minimum for the mean motion of Priamus and Patroclus. He finds the maximum of Priamus (1986.8) takes place when the planet is on a line with and between the point of oscillation and the Sun, and the minimum for Patroclus (1985.8) when the planet is on the line with and beyond the point of oscillation and the Sun. This verified his previous conclusion[2] that the two planets are diametrically opposite in their paths of oscillation.

A set of elements and an ephemeris are published in Kleine Planeten for 1920. The elements are probably Elements B, brought up to epoch 1925. An ephemeris is published in Kleine Planeten for 1921.

In A. N. Vol. 215, No. 5147, Wilkens outlines his investigation regarding the secular variations of the major axes of the orbits for the Trojan group.

REFERENCES

[1] A. N. vol. 205, p. 141. M. N. 78, p. 289.
[2] A. N. vol. 207, p. 9.
[3] A. N. vol. 207, p. 183.
[4] A. N. vol. 205, No. 4906.
[5] A. N. vol. 206, No. 4937.
[6] A. N. vol. 208, No. 4984.
[7] A. N. vol. 209, No. 5016.
[8] A. N. vol. 215, No. 5147.
[9] A. N. vol. 206, No. 4945.

TABLE 20.—*Elements (884)—Priamus (1917 CQ.)*

Letter	Epoch	M.T.	M	ω	Ω	i
			° ′ ″	° ′ ″	° ′ ″	° ′ ″
A......	1917 Sept. 24.5.........	Greenwich..	83 18 55	329 32 38	300 41 27	8 51 26
B......	1917 Sept. 24.5.........	Greenwich..	83 18 55	329 32 18	300 45 28	8 51 28
C......	1917 Sept. 24.5.........	Greenwich..	83 46 41	329 4 44	300 48 28	8 51 28
D......	1917 Sept. 24.5.........	Greenwich..	82 22 47	329 51 19	300 49 27	8 51 24
E......	1918 Oct. 29.5.........	Greenwich..	115 11 59	329 45 49	300 49 27	8 51 24
F......	1918 Oct. 29.5.........	Greenwich..	115 33 19	329 24 42	300 49 27	8 51 24

Letter	Epoch	M.T.	φ	μ	Equinox	Authority
			° ′ ″	″		
A......	1917 Sept. 24.5.......	Greenwich..	6 46 53	294.427	1917.0	Berberich
B......	1917 Sept. 24.5.......	Greenwich..	6 46 53	294.427	1925.0	Berberich
C......	1917 Sept. 24.5.......	Greenwich..	6 46 54	294.989	1925.0	Berberich-Klose
D......	1917 Sept. 24.5.......	Greenwich..	7 5 53	294.427	1925.0	Klose
E......	1918 Oct. 29.5.......	Greenwich..	7 5 59	294 5850	1925.0	Klose
F......	1918 Oct. 29.5.......	Greenwich..	7 7 34	295.0965	1925.0	Klose

TABLE 21.—*References for Observations of (884) Priamus (1917 CQ.)*

	Date	Place	Reference
1917...	September 22...	Heidelberg....	A.N. 205 p. 141, M.N. 78 p. 289* E.Z. 1917 No. 535
	September 23...	Heidelberg....	A.N. 205 p. 141; E.Z. 1917 No. 535
	September 24...	Heidelberg....	A.N. 205 p. 141; E.Z. 1917 No. 535
	September 25...	Heidelberg....	A.N. 205 p. 141; E.Z. 1917 No. 535
	September 26...	Heidelberg....	A.N. 205 p. 141; E.Z. 1917 No. 535
	October 6...	Wien........	A.N. 207 p. 149
	October 11...	Wien........	A.N. 207 p. 149
	October 13...	Wien........	A.N. 207 p. 149
	October 16...	Wien........	A.N. 207 p. 149
	October 21...	Heidelberg....	A.N. 205 p. 239 E.Z. 1917 No. 537
	November 8...	Heidelberg....	A.N. 205 p. 239 E.Z. 1918 No. 537 A.N. 206 p. 63
	December 4...	Heidelberg....	A.N. 205 p. 279 E.Z. 1918 No. 538 A.N. 206 p. 63
1917...	December 4...	Bergedorf....	A.N. 208 p. 39 E.Z. 1918 No. 559
1918...	January 2...	Bergedorf....	A N. 208 p. 39 E.Z. 1918 No. 559
	January 3...	Heidelberg....	A.N. 206 p. 63 A.N. 206 p. 15
	January 14...	Bergedorf....	A.N. 208 p. 39 E.Z. 1918 No. 559
	October 5...	Heidelberg....	A.N. 207 p. 239 E.Z. 1918 No. 555
	October 30...	Heidelberg....	A.N. 207 p. 283 E.Z. 1918 No. 557
	November 23...	Heidelberg....	A.N. 208 p. 13 E.Z. 1918 No. 558 A.N. 208 p. 167
1919...	October 21...	Heidelberg....	B.Z. 1919 No. 13 Vol. 1
1921...	January 15...	Heidelberg....	B.Z. 1921 No. 3 Vol. 3

*Discovery date.

(911) AGAMEMNON, 1919 FD.

This planet was discovered by Reinmuth[1] at Heidelberg on March 19, 1919.

The nature of its motion (Jupiter group) was noted about the same time by Palisa and Berberich.[2] The preliminary elements A by Berberich are published in A. N. Vol. 208, p. 332. A comparison between computation and observation[3] for 1919 May 20, gives $\Delta\alpha = -4^s \; \Delta\delta = -2'$.

Elements A are brought forward to mean equinox of 1925.0 and with an ephemeris are published in Kleine Planeten for 1920, p. 25 and p. 47.

An observation[4] on March 11, 1920, gives a correction to the ephemeris of $+1^m.5$ in right ascension and —18′ in declination.

An ephemeris for 1921 is given in Kleine Planeten for 1921, p. 20.

REFERENCES

[1] A. N. vol. 208, p. 231. Royal A. S. vol. 80, p. 405, 406.
[2] A. N. vol. 208, p. 331. A. N. vol. 210, p. 241.
[3] A. N. vol. 210, p. 244.
[4] B-Z der A. N. No. 13, 1920.

Elements—(911) Agamemnon (1919 FD.)

	Epoch	M.T.	M.	ω	Ω	i
A	1919 Mar. 19.5	Grw.	88°48′19″	78°46′08″	336°55′10″	21°56′50″

φ	μ	Equinox	Authority	Remarks
4°55′43″	303″.190	1919.0	Berberich.	Preliminary orbit.

TABLE 22.—*References for Observations of (911) Agamemnon (1919 FD)*

	Date	Place	Reference
1919...	March 19......	Königstuhl...	A.N. 208 p. 231*
	April 2......	Königstuhl...	A.N. 208 p. 231
	May 20......	Königstuhl...	A.N. 208 p. 347
	April 5......	Wien........	A.N. 211 p. 430
	April 6......	Wien........	A.N. 211 p. 430
	April 19......	Wien........	A.N. 211 p. 430
	April 19......	Wien........	A.N. 211 p. 430
	April 23......	Wien........	A.N. 211 p. 430
	April 25......	Wien........	A.N. 211 p. 430
	April 29......	Wien........	A.N. 211 p. 430
	May 1......	Wien........	A.N. 211 p. 430
1920...	March 11......	Bergedorf....	B. Z. 1920, No. 13

*Discovery Date.

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